BIBLIOTECA NAZ.
Vittorio Emanuele III
XXXIV
D
98
NAPOLI

34
G.
4 43+

Digitized by Google

ANALYSEOS
INFINITE PARVORUM
SIVE CALCULI DIFFERENTIALIS
ELEMENTA

ILLUSTRISSIMO, CLARISSIMOQUE VIRO

ANTONIO MORMORAJ

SACRÆ RELIGIONIS D. STEPHANI,

ET ACADEMIÆ PISANÆ AUDITORI

CONSECRATA

AUCTORE RAYNERIO BONAVENTURA MARTINI

MEDICINÆ DOCTORE, ET IN EADEM ACADEMIA

RATIONALIS PHILOSOPHIÆ PROFESSORE.

BIBLIOTECA NAZIONALE VITTORIO EMANUELE III NAPOLI

B. R. NAPOLIS

PISIS. ANNO MDCCLXI.

Ex Typographia Archiep. apud Jo: Dom. Carotti.

SUPERIORUM PERMISSU.

Digitized by Google

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]



[illegible]

[illegible]

Digitized by Google

ILLUSTRISSIMO, CLARISSIMOQUE VIRO

ANTONIO MORMORAJ

RAYNERIUS BONAVENTURA MARTINI

F. P.

QUæ ſuperioribus annis de Calculo differentiali partim collegeram, partim mecum ego meditatus fueram, quæque inter adverſaria perpetuo relinquere conſtitueram, quod ipſa per ſe nihil habeant dignum

tanto

Digitized by Google

tanto hoc litterarum splendore, & ſtudiis hominum non alienum, ea nunc repente in lucem publicis deſcripta typis educere decrevi, tuoque nomine VIR ILLUSTRISSIME conſecrare. Cum enim Tibi ſemper maxime proprium fuerit, ac ſingulare non modo litteras colere, & fovere, ſed, ex quo Tibi CÆSAR *Academiæ Piſanæ curam, moderationemque commiſit, induſtriam in eo incredibilem ponere, ut honeſtiſſima quæque præclariſſimarum artium ac litterarum ſtudia promoverentur, & Academia hæc noſtra in altiſſimo felicitatis, dignitatiſque faſtigio collocaretur, illud Academicis omnibus eſſe obnitendum videbam, ut Tibi aliqua ex parte placere poſſent, tuæque optimæ voluntati aliquo grati animi teſtimonio, ac ſignificatione reſpondere. Mecum itaque ſæpenumero reputanti, quid Tibi potiſſimum pro tua erga litteras, & litteratos homines caritate rependerem, aut quid potius eniti poſſem, quo ſtudioſum animum, cupidumque rependendi ad memoriam omnem poſteritatis declararem, occurrit animo Tibi leve hoc munuſculum conſecrare, Teque proinde labori huic meo patronum adoptare. Quanquam non eſt profecto munus, quod Tibi nunc defero; neque enim is ego ſum, qui Tibi munera dare poſſim, ſed eſt tale quidpiam, quod*

debitionem

debitionem publice profitetur, quodque, licet minimum, eſt tamen aliquid, ſi non ex commodo, & præſtantia, ſed ex viribus, & voluntate penſetur. Tu itaque, cur meus hic labor in publicam lucem proferatur, es in cauſa; Tuis ergo ſe tuebitur auſpiciis, tuique nominis amplitudine, tuoque patrocinio non invidorum modo calumnias effugiet, ſed ad aliquam etiam hominum famam, atque exiſtimationem profluet. Nec mihi nunc eſt animus tuas laudes VIR ILLUSTRISSIME recenſere, qui ſapientum præcepta non dictis perſequeris, ſed factis exprimis, & vitæ ſanctitate, quem natura ipſa finxit ad virtutem, quem honorum cumulus non extulit ad faſtum, ſed ad excellentiam virtutis traduxit, cujus mores integerrimi, ſuaviſſimique, & ad omnem humanitatis rationem temperati, cujus conſilium, gravitas, pietas, religio, aliæque harum adminiſtræ comiteſque virtutes omnium quotidie linguis prædicantur, & memoria nunquam hominum dilabentur. Quid! quod ſi id maxime conari velim, nullo modo quæ de tuis virtutibus ſe ſe offerunt celebranda cogitatione aſſequi poſſem, ne dum explicare dicendo. Satis ſuperque virtutes tuas exquiſito quodam, ac plane ſingulari genere laudum extollit recens honoris acceſſio, in

quem non ambitu, non gratia, non errore hominum obrepſiſti, ſed probitatis, & juſtitiæ commendatione, atque ea doctrinæ fama, quam non ex puriſſimis modo Sapientiæ fontibus hauſiſti, verum etiam ex præclariſſima conſuetudine, atque doctrina Parentis tui Viri longe Sanctiſſimi, ac Sapientiſſimi, cujus quidem inſtitutis, atque ſententiis Tibi a natura inſitum ingenii acumen ac robur judicii perfectum eximie eſt, & mirandum plane in modum omni occaſione reſplenduit. Quod ſupereſt, illud unum nunc Te magnopere oro, atque obſecro, ut laborem hunc qualemcumque meum hilari, ac ſerena fronte excipias, & obſervantiam meam tuæ auctoritati, tuæque beneficentiæ debitam optatiſſimo patrocinii tui honore proſequaris.

AUCTO-

Digitized by Google

AUCTORIS PRÆFATIO.

SCientia analytica in duas partes dividitur, quarum altera eſt analyſis quantitatum finitarum, ſive Carteſiana, altera, de qua nunc agimus, & quæ perfectam abſolutamque illius cognitionem deſiderat, dicitur analyſis quantitatum infinite minimarum, quippe quæ finitarum quantitatum differentias infinite parvas, & hujuſmodi differentiarum infinite minimas differentias, ſive quantitatum finitarum differentias ſecundas, tertias, quartas, & ita deinceps ſine limite ſimul comparat, earumque differentiarum relationes inveſtigat, atque ex hiſce relationibus quantitatum finitarum relationes determinat. Prima hujuſce methodi fundamenta in operibus nonnullorum vel ex ipſis veteribus Geometris præluxere. Cum enim ſolis primæ

geometriæ principiis inhærentes ad quorundam problematum ſolutionem caligarent, atque adeo ad illa conficienda longe diſſitis a communi EUCLIDIS, & APOLLONII geometria viis eſſe opus intelligerent, novas ſubinde methodos excogitarunt, quarum ope cognitiones ſuas geometricas novis inventis cumularent, & problemata altioris indaginis enodarent. Et quidem ingens ARCHIMEDES omnium primus polygona circulo, vel ſectori inſcripſit, & circumſcripſit, donec ſegmenta circuli, vel ſectoris quocumque adſignabili plano minora ſupereſſent, videlicet donec ad infinitum numerum laterum, & quoad extenſionem infinite minimorum polygona conſtituentium deveniretur, unde ipſamet inſcripta, vel circumſcripta polygona cum circulo, vel ſectore tandem convenirent. Divini hujus Geometræ veſtigiis inſiſtens Etruriæ noſtræ decus immortale GALILÆUS circulos conſideravit ut polygona ex lateribus rectis numero infinitis, & infinite parvis conſtituta, unde PASCHALIUS ſuas methodos derivavit, quibus & curvarum quadraturas, & ſoliditatem corporum ab iiſdem genitorum invenit, perimetros curvas rectificavit, hoc eſt earum

rum longitudinem detexit, figurarum, & ſolidorum centra gravitatis, atque id genus alia longe plurima determinavit. Mirifice etiam Geometra acutiſſimus CAVALLERIUS indiviſibilium methodo Geometriam exornavit; contemplatur autem vir ſummus plana ac ſuperficies perinde ac ſi ex lineis rectis numero infinitis coaleſcerent, & ſolida perinde quaſi ex planis rectis multitudine pariter infinitis conflarentur, quam poſtea methodum ad figuras planas curvilineas, & ſolida pariter curvilinea traducens celeberrimus TORRICELLIUS illas conſideravit tanquam ſi ex perimetris curvis, hæc vero ex curvis ſuperficiebus compoponerentur. Auctores tantos ſumma cum laude ſecutus ſubinde eſt longe Clariſſimus WALLISIUS, & methodo uſus non valde abſimili Cavallerianæ methodo in curvilineorum quadraturas inquirere potuit, & alia problemata enodatu difficillima facili negotio reſolvere. Hujus autem præſtantiſſimi Viri methodus in eo tota verſatur. I. Quod ſeries numerorum naturalium arithmeticæ proportionalium a zero incipiens, ſive finita ſeries ſit, ſive infinita ſit ad ſeriem totidem numerorum

 maximo

maximo æqualium ut 1. 2. II. Quod ſeries quadratorum eorundem numerorum ſit ad ſeriem totidem numerorum maximo æqualium ut 1. 3. III. Quod ſeries cuborum ſit ad ſeriem totidem numerorum maximo æqualium ut 1. 4. In quartis poteſtatibus ut 1. 5., in quintis ut 1. 7., & ſic in infinitum. Generales quoque regulas, & ad noſtras propius accedentes abſtruſiora problemata conficiendi acutiſſimus FERMATIUS invenit, ut patet ex methodo, qua de maximis, & minimis quæſtiones ſolvit, & tangentes ducit. Tum ſecuti Viri plane egregii HUDDENIUS, HEURATIUS, & BARROVIUS, quorum primus problemata de maximis, & minimis nova methodo pertractavit, novam alter commentus eſt rationem, qua lineæ curvæ in rectas commutari poſſint, hic vero novam iniit methodum tangentium ducendarum, quæ ad noſtram calculi differentialis methodum maxime accedit. Verum enim vero longe hæc omnia ſuperavit incomparabilis LEIBNITIUS, ſua methodo, quam calculum differentialem nuncupavit, & eodem fere tempore immortalis NEWTONUS methodo

illa,

Digitized by Google

illa, quam fluxionum appellat, de quibus ſupervacaneum hic eſt aliquid proferre, quandoquidem calculum infinite minimorum ſuſcipimus exponendum. Hæc autem methodus analyticæ artis complementum jure dici poteſt, & tanquam faſtigium. Multa hinc mathematica, & Phiſico-mathematica problemata, quæ in vulgari geometria propter calculorum pene dixerim infinitatem impervia erant, facillime reſolvuntur, ſcilicet problemata tangentium ad quaslibet curvas ducendarum, problemata maximorum, & minimorum, centrorum generalium, flexuum contrariorum, circulorum curvas oſculantium, curvarum catoptricarum, & dioptricarum, de quibus in hiſce inſtitutionibus agemus, problemata item quadraturæ curvarum, curvarumque perimetrorum rectificationis, nec non problema curvæ catenariæ, ſive funiculariæ, in quam catena ſuſpenſa ſe ſe componit, curvæ velariæ, in quam vela vento tumida ſinuantur, curvæ elaſticæ, quam laminæ elaterum ſe ſe evolventium æmulantur, curvæ iſochronæ, per quam grave deſcendens æqualiter æquali tempore ad datum punctum accedit, curvæ brachyſtochronæ,

næ, ſive celerrimi deſcenſus ex dato puncto ad datum punctum, regularum motuum utcumque vario celeritatis incremento, decrementove procedentium, virium centralium, centripetarum nempe, & centrifugarum, formæ navis minimæ reſiſtentiæ inter omnes, quæ in eodem fluido moventur, aliaque longe plurima, quæ ſine præſidio calculi differentialis adhuc in DEMOCRITI puteo latitarent. Ceterum in hoc opuſculo prima tantummodo pars Leibnitianæ methodi continetur, quæ quidem in eo ſita eſt, ut a quantitatibus finitis ad earum differentias infinite minimas deſcendat, unde calculus differentialis appellatur. Altera vero pars, quæ dicitur calculus integralis, quæque in eo conſiſtit, ut a quantitatibus infinite minimis ad earum tota, hoc eſt ad quantitates finitas adſurgat, ſive earundem quantitatum infinite minimarum ſummas inveniat, alio opere exponetur. Denique Lectorem benevolum obſecro, ut ſi quæ ſphalmata in hæc ſcripta irrepſiſſe deprehenderit, ea perhumaniter corrigat; Mihi enim nunc aliis ſtudiis diſtricto commodum non eſt omnia rurſus attento animo percurrere.

ANA-

ANALYSEOS INFINITE PARVORUM ELEMENTA

PARS PRIMA

DE CALCULO DIFFERENTIALI SECTIO PRIMA

CAP. I.

DEFINITIO I.

1. NOmine *Calculi differentialis* intelligitur methodus, qua differentialis, ſive infiniteſima, vel infinite parva quantitas inveniatur, ea nempe, quæ ultra quoſcumque limites decreſcere concipiatur.

SCHOL.

2. Ut autem hujuſmodi quantitatum infiniteſimarum ideam aliquam nobis effingamus, concipiamus rectam BC per axem AB curvæ AC*e* ſibi ſemper parallelam moveri, atque ita continuo creſcere, vel decreſcere, ut ad curvæ perimetrum altera ſui extremitate pertingat. *Fig.* 1. Si itaque, ducta C*c* axi AB parallela, imaginemur ordinatam BC ad rectam *be* propiuſquam concipi poſſit acceſſiſſe, ita nempe, jam ut eidem congruat, ejuſdem incrementum, vel decrementum *ce* erit differentia infinite parva ordinatæ BC, arcus vero C*e* differentia infinite parva curvæ AC*e*, & B*b* differentia infinite parva axis AB; denique areola CB*be* differentia infinite parva areæ ACB. Conſtat enim jugi hoc acceſſu ordinatæ BC ad ordinatam *be* quantitates *ce*, *b*B, *e*C, CB*be* decreſcere in infinitum, ac tandem penitus evaneſcere, ubi ordinatæ ipſæ BC, *be* inter ſe perfecte congruunt, adeoque prius infiniteſimas fieri, ſeu infinite parvas.

Digitized by Google

DEFINIT. II.

3. Quantitates *variabiles* dicuntur illæ, quæ continuo augeri poſſunt, vel minui, & poſtremis Alphabeti litteris deſignantur, nimirum x, y, z, v &c. *Conſtantes* vero dicuntur, quæ ejuſdem ſemper quantitatis perſeverant, dum aliæ immutantur, ſcilicet incrementum aliquod, vel decrementum ſuſcipiunt. Hujuſmodi autem quantitates conſtantes primis Alphabeti litteris notantur, earumque differentia nulla eſt, ſive $= 0$. Ita ex: gr: in Parabola Apolloniana A C*e*, cujus axis A B, ordinata B C, parameter A P, erit A P quantitas conſtans, quippe quæ ſemper eadem perſeverat ad quodvis parabolæ punctum referatur. Diameter vero A B, & ordinatæ B C, *be* ſunt quantitates indeterminatæ, & variabiles; nunc enim majores fiunt, nunc minores.

HYPOTH.

4. Differentiæ infinite parvæ quantitatum variabilium exprimantur per notam, ſeu litteram *d*, quam Geometræ fere omnes receperunt Anteſignano, ac Duce Cl. LEIBNITIO qui juxta Germanos analyſim hanc invenit, vel certe plurimum excoluit, atque perfecit. Nam ejuſdem ſemina in FERMATIO, BARROVIO, SLUSIO, aliiſque præluxerunt. Hinc in parabola ſi A B ſit $= x$, B C $= y$, arcus A C $= z$, erit B*b*, vel C c $= dx$, *ce* $= dy$, & arculus infinite parvus C*e* $= dz$, & ſic de reliquis differentiis.

SCHOL.

5. Angliæ Geometræ veſtigiis ingredientes Celeberrimi NEWTONI, cui laudem inventionis adſcribunt, ad deſignandas quantitatum differentias, quas ille *fluxiones*, ſive momentanea incrementa, aut decrementa magnitudinum appellabat, punctulis uti ſolent. Hinc in ſuperiori exemplo B*b*, vel C*c* $= \dot{x}$, *ce* $= \dot{y}$, C*e* $= \dot{z}$. Idem Vir Cl. calculum differentialem vocabat *methodum Fluxionum*.

PO-

POSTULAT.

6. Quantitates duæ differentia infinite minima a ſe invicem diſcrepantes pro æqualibus accipiantur: Enimvero ſi pro æqualibus habendæ non forent, adſignari poſſet ipſarum aliqua differentia; Differentia igitur earundem quantitatum non eſſet infinite minima, quod eſt contra hypotheſim; falſum ergo quod accipi non poſſint pro æqualibus.

COROLL.

7. Hinc habetur, quod ſi finitæ cuilibet quantitati vel addatur, vel ab eadem detrahatur portio aliqua ipſius infinite parva, illa ipſa quantitas ejuſdem menſuræ perſeveret; hinc $x+dx=x=x-dx$. Item quantitas $ydx+ydx+dydx$, quæ juxta vulgarem Analyſim eſt differentia rectang. xy, ut infra oſtendetur, eſt $=ydx+xdy$; ſiquidem quantitas $dydx$ eſt infinite parva relate ad ydx, & xdy, adeoque negligenda; Eſt enim ad primam ut $dy:y$, ad ſecundam vero ut $dx:x$; quæ duæ rationes ſunt infiniteſimæ; Et ſic de reliquis.

THEOREMA I.

8. In qualibet curva A C*e*, cujus ordinatæ B C, *be* ſint infinite proximæ, ſubtenſa ang. contactus, nempe recta F*e* inter tangentem curvæ T C F, & portiunculam infinite parvam C*e* intercepta eſt infinite minor ipſa infinite parva *ce* inter portiunculam curvæ C*e*, & rectam C*c* axi A B parallelam intercepta.

Ducta tangente verticis A G occurrente in D tangenti T C F, & producta *c*C axi parallela occurente tangenti ad verticem A G in puncto G, ponamus rectam D G in puncto E ita ſecari, ut ſit G E : E D in ratione finiti ad infiniteſimum, & jungatur E C, quæ producta occurret ordinatæ *be* in puncto *a* inter *c*, & *e*, erit ang. F C *a* infinite minimus; quare puncta *a*, & F ſunt infinite proxima, multoque magis puncta *a*, & *e*, quæ proinde coincidunt inter ſe; Eſt autem G E:

ED $= ca$, vel $ce : e$F; quare Fe erit infinite minor ipſa infinite parva ec, & ordinis ad hanc ſuperioris. Quoniam ergo $ec = dy$, erit Fe baſis ang. contingentiæ $= ddy$.

COROLL. I.

9. Hinc patet, quod nihil accreſcet ec per additionem ipſius eF, ſed adſumi poterit pro æquali ad ipſam eF.

COROLL. II.

10. Patet præterea, quod ducta ſubtenſa arcui curvæ Ce infinite parvo cum recta CF tangentis portiuncula conveniet, quare ipſarum altera alterius loco accipi poterit. Ex demonſtratis enim punctum F tangentis infinite proximum puncto C convenit cum puncto e curvæ ACe.

COROLL. III.

11. Potiori jure ſubtenſa ipſa pro portione curvæ infinite parva Ce, & hæc pro portione infinite parva tangentis CF poterit uſurpari, & curva quælibet adſumi ut polygonum ex lateribus rectis numero infinitis, & infinite parvis conſtitutum; tangens vero curvæ in quolibet puncto erit productio unius ex dictis lateribus infinite parvis.

COROLL. IV.

12. Hinc etiam colligitur quadrilinea BbcC, BbeC, BbFC inter ſe convenire, adeoque pro æqualibus haberi poſſe, utpote quæ differentiis infinite parvis inter ſe diſcrepant.

COROLL. V.

13. Eodem jure ſi ex determinato puncto A ducantur ſubtenſæ AC, Ae, erit ſector infinite parvus CAe pro triangulo rectilineo computandus; ſiquidem arcus Ce infinite parvus pro recta adſumi poteſt.

COROLL. VI.

14. Conſtat denique rectam eF, quæ eſt differentia ſecunda ordinatæ be, exprimere digreſſum curvæ a recta TC tangente curvam ACe in puncto C.

PROBL. I.

15. Quantitatum $x+y$, & $x-y$, ſcilicet quantitatum per additionem, aut ſubtractionem compoſitarum differentias invenire.

Quantitas variabilis x augetur differentia ſua dx, & y differentia dy; Ergo quoniam propoſita quantitas duas habere debet differentias, iiſdemque augetur, fit $x+dx+y+dy$, vel $x+dx-y-dy$. Subducatur quantitati datæ ſimul cum ejus incremento minimo quantitas $x+y$, quæritur enim differentia infinite parva datæ quantitatis, non quantitas ipſa, & ejuſdem ſimul differentia: Quare erit $x+dx+y+dy-x-y=dx+dy$, vel $x+dx-y-dy-x+y=dx-dy$.

REGULA

16. Hinc colligitur ſumi differentias quantitatum per additionem, aut ſubtractionem compoſitarum, ſi ſumatur differentia ſingulorum terminorum propoſitæ quantitatis, & eadem ſigna retineantur.

PROBL. II.

17. Quantitatis zy, nempe quantitatum ſe ſe mutuo multiplicantium differentias invenire.

Quoniam quantitas variabilis z cum ſua differentia fit $z+dz$, & variabilis y fit $y+dy$, propoſita quantitas abit in $(z+dz)\ (y+dy)=zy+zdy+ydz+dydz$. Sumatur differentia inter hanc quantitatem, & propoſitam zy, habebitur quantitas $zdy+ydz+dydz$. Atque $dydz$ eſt quantitas infinite minima relate ad alias duas $zdy+ydz$, adeoque negligenda (§ 7) ergo differentia quantitatis zy eſt $zdy+ydz$.

Aliter. Sit curva AHF, cujus axis AC, eique normaliter inſiſtat ordinata BH, quæ ad arbitrium producatur. Ex puncto A ducatur recta AE parallela rectæ BK, & per punctum H extendatur recta DHG axi AC parallela. Sit $AB=z$, quæ augeatur quantitate infinite parva $Bb=dz$, ſitque $BH=AD=y$, quæ pariter augeatur quantitate infinite parva $De=He=dy$, & ducan-

Fig. 2.

Digitized by Google

& ducantur rectæ bb, ec parallelæ rectis BH, DH. Eſt quidem differentia areæ ABH = rectang. $Bb = ydz$, & differentia areæ ADH = rectang. $Dc = zdy$; Atqui æquales quantitates æquales habent differentias, ergo differentia rectang. zy = BD = ABH + ADH eſt = $ydz + zdy$; prorſus ut ſupra.

REGULA.

18. Unde conſtat ſumi differentias quantitatum ſe ſe mutuo multiplicantium, ſi quantitatis cujuſque variabilis differentia per alias tum conſtantes, tum variabiles multiplicetur, & eorundem productorum ſumma accipiatur.

Itaque differentia quantitatis $abxz$ eſt $abxdz + abzdx$. Item differentia quantitatis xyz eſt $yzdx + xzdy + xydz$.

Demonſtratur. Fiat $xy = v$, erit $xyz = vz$, adeoque $d.\,xyz = d.\,vz = vdz + zdv$. Sed $dv = xdy + ydx$, ergo ſubſtitutis hiſce valoribus loco dv in prima differentia, & xy loco v, erit $d.\,xyz = xydz + xzdy + yzdx$. Et ſic de ceteris.

SCHOL.

19. Num. 1. Obſervandum hic eſt differentiam quantitatis zy non ſemper eſſe $= zdy + ydz$, ſed quandoque terminorum alterum affici ſigno —, ſeu fieri negativum. Profecto ſi creſcente z quantitate infinite parva dz, augeatur etiam y infinite parva quantitate dy, erit differentia ipſius $zy = zdy + ydx$, quod patet

Fig. 3 in rectang. ABCD, cujus differentia eſt rectang. infiniteſim. Bc + rectang. infiniteſim. DE, neglecto rectang. CF utpote infinities minori; nimirum ſi AB ſit $= z$, $Bb = dz$, AD $= y$, $Dd = dy$, erit differentia $ydz + zdy$, neglecta infinities minori quantitate $dzdy$.

Num. 2. Sed ſi rectang. ABCD $= zy$ inſcribatur curvæ VCO, producta AB in b, ductaque bF paral-

Fig. 4. lela ipſi BC, & completo rectang. AbFD, patet latus AD minui quantitate infinite parva dD $= dy$; quare

quare differentia rectang. ABCD erit $BbFE - DCEd$, hoc eſt $ydz - zdy$, & quidem ſi rectang. BF majus ſit altero dC, vel $zdy - ydz$ ſi rectang. dC majus ſit altero BF.

COROLL.

20. Quoniam vero $y^2 = yy$, erit $d.y^2 = d.yy = ydy + ydy = 2ydy$ (§ 18) Item quoniam $y^3 = yyy$ erit $d.y^3 = d.yyy = y^2\,dy + y^2\,dy + y^2\,dy = 3y^2\,dy$.

REGULA.

21. Hinc eruitur ſumi differentiam cujuſcumque poteſtatis perfectæ, ad quam quantitates variabiles evectæ ſunt, ſi ſumatur factum ex exponente poteſtatis hujuſce in quantitatem ipſam ad eandem poteſtatem evectam demta unitate, & per differentiam quantitatis illius variabilis multiplicetur.

Quare $d.y^m = my^{m-1}dy$, & $d.x^{m-n} = (m-n)\,x^{m-n-1}dx$. Demum $d\,(a-x)^n = -n\,(a-x)^{n-1}dx$. Demonſtratur poſtremum.

Fiat $a - x = z$, erit $(a-x)^n = z^n$, quare $d\,(a-x)^n = d.z^n = nz^{n-1}dz$. Loco z, & dz ſubſtituantur eorum valores, erit $n\,[a-x]^{n-1}.-dx$, hoc eſt $-n\,[a-x]^{n-1}dx = d\,(a-x)^n$.

Pari ratione invenietur $d\,[ay-xx]^q$, ſi fiat $ay - xx = z$; erit enim $[ay-xx]^q = z^q$; Quare $d\,(ay-xx)^q = d.z^q = qz^{q-1}dz$. Loco z, & dz ſufficiantur eorum valores, erit $q\,[ay-xx]^{q-1}(ady - 2xdx) = d\,(ay-xx)^q$.

Item

Item invenietur differentia quantitatis $ay^{m+n} + x^m (a+x)^n$. Fiat enim $a+x=z$ erit $(a+x)^n = z^n$, ergo data quantitas migrabit in $ay^{m+n} + x^m z^n$; Sed quantitatis hujuſce differentia eſt $(m+n)\, a y^{m+n-1} dy + m z^n x^{m-1} dx + n x^m z^{n-1} dz$; hinc ſi loco z, & dz ſufficiantur eorum valores, erit $(m+n)\, a y^{m+n-1} dy + n x^m [a+x]^{n-1} dx + m x^{m-1} dx\, [a+x]^n$ differentia propoſitæ quantitatis.

Non abſimili ratione invenientur differentiæ ſerierum, quas una tantum variabilis ingreditur. Sit ex: gr: invenienda differentia ſeriei $y^m\, [a + by^n + cy^{2n} + fy^{3n} + \&c.]^q$. Fiat $a + by^n + cy^{2n} + fy^{3n} + \&c. = z$, erit $[a + by^n + cy^{2n} + fy^{3n} + \&c.]^q = z^q$, & $y^m (a + by^n + cy^{2n} + fy^{3n} + \&c.)^q = y^m z^q$, cujus differentia eſt $m z^q y^{m-1} dy + q y^m z^{q-1} dz$, ſive $m y^{m-1} z . z^{q-1} dy + q y\, dz . y^{m-1} z^{q-1}$, hoc eſt $[m z\, dy + q y\, dz]\, y^{m-1} z^{q-1}$. Fiant neceſſariæ ſubſtitutiones, eritque

$$\left[\begin{array}{l} ma + mby^n + mcy^{2n} + mfy^{3n} + \&c. \\ + nqby^n + 2nqcy^{2n} + 3nqfy^{3n} + \&c. \end{array}\right] y^{m-1} z^{q-1} dy$$

quæſita differentia.

Eodem plane modo invenietur differentia plurium ſerierum ſe ſe invicem multiplicantium, ut patet.

LEMMA I.

LEMMA I.

22. Fractio $\frac{1}{x^n}$ eſt $= x^{-n}$.

Conſtet ex Algebra Finitorum $\frac{x^m}{x^n} = x^{m-n}$.

Ergo ſi $m = n$, erit $\frac{x^m}{x^n} = 1$; numerator enim æqualis eſt denominatori, & $x^{m-n} = x^0$; hinc $1 = x^0$; quare ſi m ponatur $= 0$, erit $\frac{x^m}{x^n} = \frac{x^0}{x^n} = \frac{1}{x^n} = x^{0-n} = x^{-n}$.

LEMMA II.

23. Quantitas $\sqrt{x}$ eſt $= x^{\frac{1}{2}}$.

Nam $\sqrt{x} . \sqrt{x} = x$; præterea $x = x^{\frac{1}{2}} . x^{\frac{1}{2}}$; Ergo $\sqrt{x} . \sqrt{x} = x^{\frac{1}{2}} . x^{\frac{2}{2}}$: Quare ſi tota inter ſe æqualia per eorundem dimidia dividantur, erunt quoti inter ſe æquales. Sed $\frac{\sqrt{x} . \sqrt{x}}{\sqrt{x}} = \sqrt{x}$, & $\frac{x^{\frac{1}{2}} . x^{\frac{1}{2}}}{x^{\frac{1}{2}}} = x^{\frac{1}{2}}$, ergo $\sqrt{x} = x^{\frac{1}{2}}$.

Eodemque modo demonſtrabitur $\sqrt[3]{x} = x^{\frac{1}{3}}$, $\sqrt[5]{x^3} = x^{\frac{3}{5}}$, & generatim $\sqrt[m]{x^n} = x^{\frac{n}{m}}$.

24. Quoniam vero $x^{-\frac{n}{m}} = \frac{1}{x^{\frac{n}{m}}}$, & $x^{\frac{n}{m}} = \sqrt{x^{n}}$, erit $\frac{1}{\sqrt[m]{x^{n}}} = \frac{1}{x^{\frac{n}{m}}} = x^{-\frac{n}{m}}$

PROBL. III.

25. Quantitatum radicalium differentias invenire.

I. Invenienda ſit differentia $\sqrt{(xy+y^2+x^2)}$. Fiat hæc quantitas radicalis $= z$, erit $xy+y^2+x^2 = z^2$, & $ydx+xdy+2ydy+2xdx = 2zdz$, ergo $\frac{ydx+xdy+2ydy+2xdx}{2z} = dz$; hoc eſt ſi loco z ſufficiatur ejus valor, $\frac{ydx+xdy+2ydy+2xdx}{2\sqrt{(xy+y^2+x^2)}} = dz$.

II. Invenienda ſit differentia $\sqrt[3]{(ay+y^2)}$. Fiat hæc $= z$, erit $ay+y^2 = z^3$, & $ady+2ydy = 3z^2dz$, $\frac{ady+2ydy}{3zz} = dz$, ſive $\frac{ady+2ydy}{3\sqrt[3]{(ay+y^2)^2}} = dz$.

III. Sumenda ſit differentia $\sqrt{[ay+y^2+\sqrt{[b^3+bxy^2]}]}$. Ponatur hæc quantitas $= z$, erit $ay\ \ y^2+\sqrt{[b^3+bxy^2]} = z^2$, ergo $ady+2ydy+\frac{by^2dx+2bxydy}{2\sqrt{[b^3+bxy^2]}} = 2zdz$ (*num.* I.), &

$\frac{a\,dy + 2y\,dy}{2z} + \frac{by^2\,dx + 2bxy\,dy}{2z \cdot 2\sqrt{[b^3 + bxy^2]}} = dz$, ſive

$$\frac{a\,dy + 2y\,dy}{2\sqrt{(ay + y^2 + \sqrt{(b^3 + bxy^2)})}} + \frac{by^2\,dx + 2bxy\,dy}{2\sqrt{[ay + y^2\sqrt{[b^3 + bxy^2]}]}\; 2\sqrt{[b^3 + bxy^2]}} = dz$$

IV. Sit invenienda differentia $\sqrt[n]{z}$. Fiat $\sqrt[n]{z} = y$, erit $z = y^n$, $dz = ny^{n-1}dy$, $\frac{dz}{ny^{n-1}} = dy$, ſive $\frac{y\,dz}{ny^n} = dy$, hoc eſt $\frac{dz\,\sqrt[n]{z}}{nz} = dy$. Sed $\frac{\sqrt[n]{z}}{nz} = \frac{\sqrt[n]{z}}{n\sqrt[n]{z^n}} = \frac{1}{n\sqrt[n]{z^{n-1}}}$: Ergo $\frac{dz}{n\sqrt[n]{z^{n-1}}} = dy$.

COROLL.

26. Quoniam vero $\sqrt[n]{z} = z^{\frac{1}{n}}$ (§ 23) erit $d.z^{\frac{1}{n}} = d.\sqrt[n]{z}$, nempe $= \frac{dz}{n\sqrt[n]{z^{n-1}}}$; Quod ſic ulterius demonſtrari poteſt. Fiat $z^{\frac{1}{n}} = y$, erit $z = y^n$, $dz =$

$ny^{n-1}dy$, $\frac{dz}{ny^{n-1}} = dy$, ſive $\frac{ydz}{ny^{n}} = dy$. Fiant opportunæ ſubſtitutiones, erit $\frac{dz\sqrt[n]{z}}{n\sqrt[n]{z^{n}}} = dy$, hoc eſt $\frac{dz}{n\sqrt[n]{z^{n-1}}} = dy$, ut ante.

V. Invenienda ſit differentia $\sqrt[m]{y^{n}}$. Fiat $\sqrt[m]{y^{n}} = z$, erit $y^{n} = z^{m}$, $ny^{n-1}dy = mz^{m-1}dz$, & $\frac{ny^{n-1}dy}{mz^{m-1}} = dz$, ſive $\frac{nzy^{n-1}dy}{mz^{m}} = dz$, hoc eſt $\frac{ny^{n-1}dy\sqrt[m]{y^{n}}}{my^{n}} = dz$, vel $\frac{n}{m}y^{\frac{n}{m}-1}dy = dz$, ſive demum $\frac{n}{m}y^{\frac{n-m}{m}}dy = dz$.

COROLL.

27\. Cum autem $\sqrt[m]{y^{n}} = y^{\frac{n}{m}}$, erit $d.y^{\frac{n}{m}} = d.\sqrt[m]{y^{n}}$, quod demonſtratur ut ſupra.

PROBL. IV.

28\. Quantitatum ſe ſe mutuo dividentium differentias invenire.

I. Sumenda ſit differentia quantitatis $\frac{x}{y}$. Ponatur $\frac{x}{y} = z$, erit $x = yz$, $dx = ydz + zdy$, & $dx - zdy = ydz$

Digitized by Google

$= y\,dz$, $\frac{dx - z\,dy}{y} = dz$, hoc eſt $\frac{dx}{y} - \frac{x\,dy}{yy} = dz$, ſive $\frac{y\,dx - x\,dy}{yy} = dz$.

Aliter. Nam $\frac{x}{y} = xy^{-1}$. Sed $d.xy^{-1} = y^{-1}$

$dx - xy^{-2}dy = \frac{dx}{y} - \frac{x\,dy}{yy} = \frac{y\,dx - x\,dy}{yy}$;

Ergo patet propoſitum.

II. Accipienda ſit differentia $\frac{\sqrt[3]{(ay + y^2)}}{\sqrt{(xy + y^2)}}$. Fiat

$\sqrt[3]{(ay + y^2)} = z$, & $\sqrt{[xy + y^2]} = v$; erit

$\frac{\sqrt[3]{(ay + y^2)}}{\sqrt{(xy + y^2)}} = \frac{z}{v}$, & $d\frac{\sqrt[3]{[ay + yy]}}{\sqrt{(xy + yy)}} = \frac{v\,dz - z\,dv}{v^2}$.

Reſtituantur in locum v, & z eorundem valores, erit

$\frac{\sqrt{(xy + y^2)}\,dz - \sqrt[3]{[ay + y^2]}\,dv}{xy + y^2} = \frac{v\,dz - z\,dv}{v^2}$.

Sed $dz = \frac{a\,dy + 2\,y\,dy}{3\sqrt[3]{(ay + yy)}}$, & $dv = \frac{x\,dy + y\,dx + 2\,y\,dy}{2\sqrt{(xy + yy)}}$; Quare ſi loco dz, & dv ſufficiantur eorum valores nunc inventi, erit

$\frac{\sqrt{(xy + yy)}(a\,dy + 2\,y\,dy)}{(xy + yy)\,3\sqrt[3]{(xy + yy)^2}} -$

$\frac{\sqrt[3]{[ay+y^2]}\,[x\,dy+y\,dx+2y\,dy]}{(xy+yy)\,2\sqrt{(xy+yy)}} = \frac{v\,dz-z\,dv}{vv}$, quæ eſt differentia fractionis $\frac{z}{v}$, ſive $\frac{\sqrt[3]{(ay+yy)}}{\sqrt{(xy+yy)}}$. Conſtat ergo propoſitum.

III. Sumenda ſit differentia quantitatis $\frac{1}{y^m} = y^{-m}$. Ponatur $\frac{1}{y^m} = z$, erit $1 = zy^m$, $0 = y^m\,dz + mzy^{m-1}\,dy$, hoc eſt $-mzy^{m-1}\,dy = y^m\,dz$, & $\frac{-mzy^{m-1}\,dy}{y^m} = dz$, ſive $-\frac{my^{m-1}\,dy}{y^{2m}} = dz$, vel demum $-my^{-m-1}\,dy = dz$.

IV. Sit invenienda differentia $\frac{[a+y]^m}{(b+y)^n}$. Sit $\frac{a+y}{b+y} = \frac{z}{v}$, erit $\frac{[a+y]^m}{b+y]^n} = \frac{z^m}{v^n}$; quare $d\,\frac{(a+y)^m}{[b+y]^n} = \frac{d.z^m}{v^n} = \frac{mv^n z^{m-1}\,dz - nz^m v^{n-1}\,dv}{v^{2n}}$, hoc eſt ſuffectis valoribus quantitatum v^n, v^{n-1}, dv, z^{m-1},

Digitized by Google

$z^m, dz,$

$$\frac{m[b+y]^n [a+y]^{m-1} dy - n(a+y)^m (b+y)^{n-1} dy}{[b+y]^{2n}}$$

nempe inſtituta diviſione per $(b+y)^{n-1} =$

$$\frac{m[b+y][a+y]^{m-1} dy - n[a+y]^m dy.}{[b+y]^{n-1}}$$

V. Sit ſumenda differentia quantitatis $\frac{1}{\sqrt[m]{y^n}} = y^{-\frac{n}{m}}$. Fiat $\frac{1}{\sqrt[m]{y^n}} = z$, erit $1 = z\sqrt[m]{y^n}$, ergo $1 =$ $z^m y^n, 0 = m y^n z^{m-1} dz + n z^m y^{n-1} dy$, $-\frac{n z^m y^{n-1} dy}{m y^n z^{m-1}} = dz$, hoc eſt $-\frac{n y^{-1} dy}{m\sqrt[m]{y^n}} = -\frac{n dy}{m y \sqrt[m]{y^n}} = \frac{-n dy.}{m\sqrt[m]{y^{n+m}}}$

REGULA

29. Unde conſtat ſumi differentias quantitatum ſe ſe dividentium, vel fractionum, ſi differentia numeratoris in denominatorem ducta ſubtrahatur. Ex differentia denominatoris per denominatorem multiplicata, & reſiduum per quadratum denominatoris dividatur.

CAP.

Digitized by Google

CAP. II.

De uſu Calculi differentialis in Tangentibus normalibus, & aſymptotis Curvarum determinandis.

DEFINITIO III.

30. T*Angens* Curvæ dicitur recta TM, quæ ita tangit quemvis curvæ arcum AM in M, ut nulla recta duci poſſit per M inter rectam TM, & arcum AM, vel intra angulum contactus AMT ab iiſdem efformatum.

COROLL.

Fig. 5. 31. Si producta recta TM in C ſit MC tangens arcus MG, quæ eſt continuatio ipſius MA, tunc arcus AMG habet continuam curvaturam in M.

DEFINIT. IV.

32. *Coordinatæ* vocantur duæ indeterminatæ datæ naturæ, ex quarum poſitione data ſingula quæſitæ lineæ puncta ſufficienter determinantur.

PROBL. V.

33. Invenire valorem ſubtangentis in qualibet curva algebraica in quovis Coordinatarum angulo.

Sit curva A M r, cujus abſciſſa AP dicatur x, ordinata PM y, quæritur ejus tangens MT, vel ſubtangens PT. Intelligatur recta pr ducta infinite proxima alteri MP, & ex puncto M agatur MR parallela diametro AP ſecans pr in R. Erit $Pp = dx$, $Rr = dy$. Cum autem triang. MRr non differat a rectilineo, ſiquidem arculus Mr infinite parvus pro recta adſumi poteſt, triang. TPM, MRr erunt ſimilia, quare $rR : RM = MP : PT$, $dy : dx = y : \frac{y\,dx}{dy} =$ ſubtangenti curvæ, quæ cum generatim accepta fuerit, generatim etiam ejuſdem inventa fuit ſubtangens. Verum ſi ex data curvæ æquatione in formula generali $\frac{y\,dx}{dy}$ valor ipſius dx ſubſti- tuatur,

tuatur, quantitates infinite parvæ, ſeu differentiales evaneſcent, & ſubtangentis, adeoque & tangentis valor in quantitatibus finitis, & communibus obtinebitur.

COROLL. I.

34. Quoniam æquatio ad parabolam Apollonianam eſt $yy = ax$, erit ſumtis differentiis, $2ydy = adx$, & $\frac{2ydy}{a} = dx$; quare ſi in formula ſubtangentis generali loco dx ſubſtituatur ejus valor $\frac{2ydy}{a}$; erit ſubtangens

$$PT = \frac{2yydy}{ady} = \frac{2yy}{a} = \frac{2ax}{a} = 2x.$$

COROLL. II.

35. Æquatio ad parabolas cujuſcumque gradus eſt $y^{m+n} = a^m x^n$; erit ergo $(m+n)y^{m+n-1}dy = na^m x^{n-1}dx$, & $\frac{(m+n)y^{m+n-1}dy}{na^m x^{n-1}} = dx$; quare ſubtangens $\frac{ydx}{dy} = \frac{(m+n)y^{m+n}}{na^m x^{n-1}} = \frac{(m+n)}{n}x$.

Hinc ſi $m = 2$, $n = 1$, erit ſubtangens parabolæ cubicæ $= 3x$; si vero $m = 1$, $n = 2$, erit ſubtangens $= \frac{3}{2}x$. Et ſic de ceteris.

COROLL. III.

36. Ad hyperbolas cujuſcumque gradus æquatio eſt $ay^{m+n} = bx^m(a+x)^n$, ergo $(m+n)ay^{m+n-1}dy = m(a+x)^n bx^{m-1}dx + nbx^m(a+x)^{n-1}dx$; hinc

hinc $\frac{y\,dx}{dy} = \frac{(m+n)\,a\,y^{m+n}}{m\,(a+x)^n\,bx^{m-1} + n\,b\,x^m\,[a+x]^{n-1}}$

$= \frac{(m+n)\,b\,x^m\,(a+x)^n}{m\,(a+x)^n\,bx^{m-1} + n\,b\,x^m\,(a+x)^{n-1}} =$

$\frac{[m+n]\;[ax+xx]}{ma+mx+nx}$. Si $m = 1$, $n = 1$, erit $\frac{y\,dx}{dy} =$ $2\,[ax+xx]$, quæ eſt ſubtangens ad hyperbolam quadraticam. Et de ceteris.

COROLL. IV.

37. Eadem ferme ratione ex data æquatione $ay^{m+n} = bx^m\,(a-x)^n$ ad Ellipſes cujuſcumque gradus invenietur earundem ſubtangens $= \frac{[m+n]\,(ax+x^2)}{ma-mx-nx}$.

COROLL. V.

Sit æquatio ad hyperbolas aſymptoticas cujuslibet gradus $a^{m+n} = x^m\,y^n$, erit $0 = my^n\,x^{m-1}\,dx$ $+ nx^m\,y^{n-1}\,dy$, & $-nx^m\,y^{n-1}\,dy = my^n\,x^{m-1}\,dx$, nempe $\frac{-nx^m\,y^{n-1}\,dy}{ym^n\,x^{m-1}} = dx$; hinc $\frac{y\,dx}{dy} =$

$\frac{-nx^m\,y^n}{my^n\,x^{m-1}} = -\frac{nx}{m}$.

Patet autem ex hoc valore ſubtangentis negativo ſubtangentem ipſam in partem averſam originis abſciſſæ ſumendam eſſe. Hunc vero valorem fieri negativum ulterius

Digitized by Google

ulterius conſtat. Enimvero differentia ipſius $x^m y^n$ erat $m y^n x^{m-1} dx - n x^m y^{n-1} dy$; Creſcente ſiquidem x decreſcit y. [§. 19. *num.* 2.]

COROLL. VI.

38. Sit æquatio $\frac{ccx}{a+x} = yy$ ad curvam, quæ adducitur a Cl: GUIDONE GRANDIO prop. 85. de Reſiſtentia Solidorum velut hyperbola ſecundi generis ab immortali NEWTONO indicata, erit $2ydy = \frac{accdx}{(a+x)^2}$, & $2yydy = \frac{accydx}{[a+x]^2}$, hoc eſt $\frac{ydx}{dy} = \frac{2yy[a+x]^2}{ac^2} = \frac{2ccx[a+x]^2}{ac^2(a+x)} = \frac{2ax+2xx}{a}$.

COROLL. VII.

39. Item inveniri poterit ſubtangens in curva EDF, in qua ſemper inclinatis AD, CD, demiſſaque normali DB ſit ſemper $DC^2 =$ rectang: A.DB. *Fig. 6.*

Poſita enim recta $AC = a$, $CB = x$, $DB = y$ erit ejuſdem æquatio $yy = \frac{x^4}{aa - 2ax - xx}$, vel $aayy - 2axyy - xxyy = x^4$, quare $2aaydy - 4axydy - 2xxydy = 4x^3dx + 2ayydx + 2yyxdx$, hinc

$$\frac{ydx}{dy} = \frac{aayy - 2axyy - xxyy}{2x^3 + ayy + yyx} = \frac{x^4}{2x^3 + ayy + yyx}.$$

COROLL. VIII.

40. Si fuerit æquatio ad verſoriam A G H genitam a ſemicirculo A C B per ramos B C F ad tangentem uſque A F protenſos, & rectam F G, quæ ducatur axi A B parallela, & occurrat ordinatæ D C productæ in G, quæ æquatio, poſita $AB = a$, $AD = x$, $DG = y$, eſt $= yy = \frac{aax}{a-x}$, ut patet; eſt enim $AB : AF = DG = BD : DC = DC : DA$; vel $y = \frac{a\sqrt{x}}{\sqrt{z}}$, poſita $a - x = z$;

Fig. 7.

erit $dy = \frac{azdx - axdz}{2z\sqrt{xz}}$, & $\frac{ydz}{dy} = \frac{2zxdx}{zdx - xdz}$; ſed $dz = -dx$, ergo in valore ſubtangentis poſito $-dx$ loco dz, erit $\frac{ydx}{dy} = \frac{2zx}{z+x}$; atqui $z = a - x$, quare $\frac{ydx}{dy} = \frac{2x(a-x)}{a} = DT$.

Ubi punctum G cadit in vertice A, quoniam tunc $a - x = a$ fiet ſubtangens $= 2x$.

COROLL. IX.

41. Si fuerit Ciſſois B E, cujus circulus genitor A M B; cum ſit $DE^2 : DB^2 = ED : DM = BD : DA$, erit, poſita $DE = y$, $CA = a$, $DB = x$, $y^2 : x^2 = x : 2a - x$, unde fiet æquatio $x^3 = 2ayy - xyy$; erit igitur $3xxdx = 4aydy - 2xydy - yydx$, ſive $\frac{4aydy - 2xydy}{3xx + yy} = dx$, quo ſuffecto in valore ſubtangentis $\frac{ydx}{dy}$, fiet ſubtangens $= \frac{4ayy - 2xyy}{3xx + yy}$. Ponatur etiam $DM = v$, erit $xx = vy$, quare ſubtan-

Fig. 8.

gens $= \frac{4ay - 2xy}{3v + y}$, hoc eſt ut ED + 3DM : ED = dupla AD : Dt ſubtangentem, ſive ut BD + 3 DA : DB = dupla DA : Dt, & alternando BD + 3DA : 2DA = DB : Dt, vel $\frac{1}{2}$ BD + $\frac{3}{2}$DA, hoc eſt $\frac{1}{2}$ BD + $\frac{1}{2}$DA + $\frac{2}{2}$ DA, ſive $\frac{1}{2}$ BA + DA : DA = DB : Dt, poſitaque AT = $a = \frac{1}{2}$ BA, erit TD : DA = BD : Dt, & TDt = BDA = DM^2; hinc juncta TM, factoque ang. recto TMt determinabitur ſubtangens Dt.

Si ordinata DE tranſeat per centrum circuli, ita ut puncta C, t coincidant, erit ut TD dupla DA, ita DB dupla Dt, ſunt enim proportionales, & Dt = $\frac{1}{2}$ radii.

PROBL. VI.

42. Invenire ſubtangentem in Conchoide NICOMEDIS.

Eſto Conchois ATE, cujus polus P, regula BF, & conſtans BA, vel HT; quæritur curvæ tangens ad punctum T. Ducatur Pbt infinite proxima ipſi PT, & centro P intervallis Pb, Pt deſcribantur arculi *bo*, *tr*, qui ob indefinitam parvitatem ſuam congruent normalibus *bo*, *tr* ex punctis *b*, & *r* in PT demiſſis, erectaque PK normali ad PT, ponatur ducta tangens TK occurrens in K rectæ PK. Sit AB = HT = a, PB = b, PH = y, BH = x, erit Ho = dy, TP = a + y, proindeque Tr = dy. Jam vero triang: HBP, Hbo ſunt ſimilia, ergo HB : BP = Ho : ob, x : b = dy : $\frac{bdy}{x}$. Præterea ob ſimilitud. triang. Pob, Prt eſt Pb : Pt = bo : tr, y : a + y = $\frac{bdy}{x}$: $\frac{abdy + bydy}{xy}$. *Fig. 9.*

Denique

Denique ob ſimilitud. triang. Trt, TPK eſt $Tr : rt = TP : PK$, $dy : \frac{abdy + bydy}{xy} = a + y :$

$\frac{aabdy + 2abydy + byydy}{xydy} = \frac{b[a+y]^2}{xy}$; Quæ eſt quæſita ſubtangens.

MONITUM.

43. Hujuſmodi methodum ducendi tangentem ad Conchoidem, quæ methodus, ut patet, ab æquatione curvæ minime profluit, & qua in curvis aliis uti poſſumus, ideo attulimus, ut conſtet, quomodo ducenda ſit tangens in curvis illis, in quibus ob æquationem nimis prolixam moleſtior, & implexior calculus evaſiſſet, & ſubtangens prodiiſſet inelegantior.

PROBL. VII.

44. Sit recta A B, quæ dividatur biſariam in puncto C, eique in punctis C, A normaliter inſiſtant rectæ FADE, QCHO; Ex puncto B ductis rectis BHD, BOE, quæ ſecent rectas AE, CO fiat HK= HC, itemque HK = ipſi HC, & ducta pariter recta BRF, fiat RQ = QC, nec non Qr = eidem rectæ QC, & ſic ſemper, deſcribetur curva KCrBkCR, cujus quæritur tangens.

Hujuſmodi curvam ſummus Geometra TORRICELLIUS vocat Pteroidem.

Ponatur quæſita tangens KT, & ex puncto B extendatur recta TBt normalis ipſi BK, quæ concurrat cum tangente, ductaque recta BL infinite proxima rectæ BK, centro B intervallis BK, BH deſcribantur arculi infinite par-
Fig. 10. vi KM, HN. Ducantur præterea ordinatæ KG, kg normales rectæ BA, & ponantur $BC = a$, $CH = HK = x$, $BK = y$, erit $HO = dx$, $ML = dy$, & $BO = \sqrt{[a^2 + x^2]}$. Et quoniam ſimilia ſunt triang:

BCO,

BCO, ONH, erit BO : BC = HO : HN, $V[aa+xx] : a = dx : \frac{adx}{V[aa+xx]} = HN$. Item ob ſimilitud. triang. HBN, KBM erit HB : BK = HN : KM, $V(aa+xx) : y = \frac{adx}{V[a^2+x^2]} : \frac{aydx}{a^2+x^2} = KM$.

Denique ob ſimil. triang. KLM, TLB eſt LM : MK = LB : LT, $dy : \frac{aydx}{a^2+x^2} = y : \frac{ayydx}{[aa+xx]dy} = BT$.

Eſt autem $V(aa+xx) = y \pm x$, & $\frac{xdx}{V[aa+xx]} = dy - dx$ in majori, vel $= dx - dy$ in minori; dum enim creſcit CH=HK fit minor ipſa Bk, quare ejus differentia dy minuitur creſcente dx. Hinc $dy = dx \pm \frac{xdx}{V(aa+xx)} = \frac{dx\,V[aa+xx] \pm x\,dx}{V(a^2+x^2)}$. Sed $y = V(aa+xx) \pm x$; quare $dy = \frac{ydx}{(Vaa+xx)}$, & $dx = \frac{dy\,V(aa+xx)}{y}$. Nunc ſi in $BT = \frac{ayydx}{(aa+xx)dy}$ loco dx ſufficiatur valor $\frac{dy\,V[aa+xx]}{y}$ fiet

$$\frac{ayydx}{[aa+xx]dx} = \frac{ayydy\,V(aa+xx)}{ydy(aa+xx)} = \frac{ay}{V[aa+xx]}$$

= BT, quæ eſt quæſita ſubtangens.

Quoniam vero HB : BC = KB : GB, $V[a^2+x^2] : a = y : \frac{ay}{V(aa+xx)}$, erit BG = BT; hinc ſi BT normalis ad BK fiat = BG, & jungantur

puncta

puncta T, K, erit TK tangens curvæ, idemque præſtandum in parte ejuſdem curvæ inferiori.

PROBL. VIII.

45. Sit ang. rectus BAC, & latus AB ſit indefinite productum. Ex puncto fixo C ſumto in latere
Fig. 11. AC ductis CE, CB utcumque inclinatis ad latus AB, fiat $CF = AE$, $CH = AB$, & ſic ſemper, quæritur tangens curvæ per puncta ſic inventa tranſeuntis.

Fiant infinite proximæ CE, CB, & ex puncto fixo C tanquam centro intervallis CE, CF deſcribantur arculi infinite parvi EG, FD, poſitaque in puncto C recta CT normali ad CB quæ occurret tangenti quæſitæ in puncto T, dicatur $AC = a$, $AE = CF = z$, erit $BC = \sqrt{aa+zz}$ hinc $BG = \frac{zdz}{\sqrt{aa+zz}}$; Cum autem ſimilia ſint triang. BGE, BAC, itemque ECF, FCD, erit $BA : AC = BG : GE$, $z : a = \frac{zdz}{\sqrt{aa+zz}} : \frac{adz}{\sqrt{aa+zz}} = GE$, & $EC : CF = EG : FD$, $\sqrt{a^2+z^2} : z = \frac{adz}{\sqrt{aa+zz}} : \frac{azdz}{aa+zz} = FD$. Denique ob ſimilitud. triang. HDF, FCT eſt $HD : DF = FC : CT$, $dz : \frac{azdz}{aa+zz} = z : \frac{azzdz}{dz(aa+zz)} = \frac{azz}{aa+zz}$, qui eſt valor ſubtangentis CT.

SCHOL.

46. Facile autem determinatur hujus curvæ ſubtangens ex ejuſdem æquatione: Eſt enim, poſita $CA = a$, $CM = LF = x$, $MF = CL = y$, $CF = z$, $aaxx = y^4 + xxyy$; quare $MP = \frac{ydx}{dy} = \frac{2y^4 + xxyy}{aax - yyx}$,

ſive

ſive loco ax poſita yz, $MP = \frac{2y^3 + xxy}{az - yx}$.

PROBL. IX.

47. Sit quævis curva ABC, cujus nota ſit tangens in quolibet puncto C, ex eaque fiat alia curva AFG, cujus ordinatæ DF, EG ſint ſemper æquales reſpective ſubtenſis AB, AC, quæritur tangens hujuſmodi curvæ AFG.

Fig. 12.

Sint AB, AC infinite proximæ, & ex puncto A erecta ad CA normalis AH conveniat cum tangente CA in puncto H, deſcriptoque arculo BI centro A intervallo AB, & ductis BK, HS ipſi AE parallelis dicatur $AC = EG = y$, $AD = x$ $AH = z$, eritque $DE = BK = FM = dx$, $CI = GM = dy$. Et quoniam ſimilia ſunt triang. HSC, BKC eſt HS : BK = HC : CB = HA : BI [ob ſimilitud. triang. HCA, BCI] quare permutando erit HS : HA = BK : BI, nempe $z : v = dx : \frac{vdx}{z} = BI$. Sed AC : CI = AH : BI, & rurſus permutando AC : AH = CI : BI, quare $y : v = dy : \frac{vdy}{y} = BI$. Hinc $\frac{vdx}{z} = \frac{vdy}{y}$, adeoque $dx = \frac{vzdy}{vy} = \frac{zdy}{y}$. Suffecto igitur in formula ſubtangentis generali $\frac{ydx}{dy}$ loco dx ejus valore $\frac{zdy}{y}$, erit quæſita ſubtangens $ET = \frac{yzdy}{ydy} = z$. Sed $z = HS$, ergo ducta HT parallela DB, erit ET quæſita ſubtangens.

Aliter ducta HT parallela ordinatæ DF, erit DT ſubtangens curvæ AFG, nempe $\frac{ydx}{dy}$. Nam eſt $CA : CI = AH : BI$, $y : dy = v : \frac{vdy}{y}$, & BI:

$AH = BC : HC = BK : HS$, adeoque $BI : AH = BK : HS$, nempe $\frac{vdy}{y} : v = dx : \frac{vydx}{vdy} = \frac{ydx}{dy}$.

COROLL.

48. Patet autem, quod pro varietate aſſumtæ curvæ ABC alia, atque alia curvæ AFG ſpecies orietur; Et quidem ſi curva genitrix ABC fuerit arcus circuli, curva genita erit parabola; si curva genitrix fuerit parabola, hyperbola, vel Ellipſis circa majorem axem, curva genita fiet hyperbola æquilatera; Et ſic de ceteris.

Antecedens problema aliter rurſus ad abundantiorem ſcientiam enodare.

49. Sint AB, AC infinite proximæ, & arculus BI. Ex vertice A agatur indefinita AVP, normalis axi AD, & ex puncto R, in quo cognita prioris curvæ tangens concurrit cum recta AVP demittantur RO normalis ad AC, & RQ normalis ad FC, & cetera ut ſupra. Sit autem $RO = v$, $OC = z$, quæſita vero tangens ponatur Ft, quæ ut determinetur, patet quod ducta GP parallela axi AD concurrente in P cum indefinita AP, inveniri debeat valor ſubtangentis Pt. Eſt autem $RQ : RO = BK : BI$, $x : v = dx : \frac{vdx}{x} = BI$. Præterea eſt $RO : OC = BI : IC$, $v : z = \frac{vdx}{x} : \frac{zdx}{x} = dy$; hinc $\frac{xdy}{dx} = z$. Denique eſt $GN : FN = GP : Pt$, $dx : dy = x : \frac{xdy}{dx} = z = Pt = OC$.

COROLL. I.

50. Ex hujuſce problematis ſolutione alia conſtat methodus, qua curvarum tangentes determinari poſſint. Sint itaque in qualibet curva Atr infinite proximæ Pt, pr, & tR parallela axi AP, atque ex puncto A, quod eſt vertex curvæ agatur indefinita *Fig.* 13

AID

Digitized by Google

A I D parallela ordinatis, quam ſecet tangens tI in puncto I. Ex puncto r erigatur recta rD parallela eidem axi A P, quæ occurrat rectæ A D in puncto D. Patet inveniendum eſſe valorem rectæ DI, quam licebit appellare *ſubtangentem*, ad hoc ut tangens ipſa tI determinari poſſit. Erit igitur $rq : qt = rD : DI$, $dx : dy = x : \frac{xdy}{dx} = DI$; Quare ſi ex data curvæ æquatione in hac generali ſubtangentis formula $\frac{xdy}{dx}$ valor ipſius dx ſufficiatur, ſubtangens ipſa determinabitur.

COROLL. II.

51. Quoniam vero in parabola $2ydy = adx$, erit $\frac{xdy}{dx} = \frac{ax}{2y} = \frac{yy}{2y} = \frac{1}{2}y$.

COROLL. III.

52. In circulo eſt $2ydy = adx - 2xdx$, hinc $\frac{xdy}{dx} = \frac{ax - 2xx}{2y} = y - \frac{ax}{2y}$.

COROLL. IV.

53. In Ellipſi vero cum ſit $2ydy = abdx - 2bxdx$, erit $\frac{xdy}{dx} = \frac{abx - 2bxx}{2ay} = y - \frac{bx}{2y}$. Et ſic de tangentibus aliarum curvarum.

PROBL. X.

54. Eſto curva A B b cujus nota ſit tangens ad quodlibet punctum B, & ſit alia curva A C c ejus naturæ ut differentia cf ſit ad differentiam bg ut $cd : bd$ quæritur hujus curvæ tangens.

Fig. 14. Sit $AD = x$, $BD = y$, $CD = z$, erit $Dd = dx$, $cf = dz$, $bg = dy$. Jam vero ſubtangens generalis curvæ ABb eſt $\frac{ydx}{dy}$, & ſubtangens curvæ ACc eſt $\frac{zdx}{dz}$, quare ſubtangens prioris curvæ erit ad ſubtangentem ſecundæ ut $\frac{y}{dy} : \frac{z}{dz}$. Sed ex natura curvæ $dz : dy = z : y$, quare $ydz = zdy$, & $dz = \frac{zdy}{y}$. Hinc ſi in $\frac{z}{dz}$ loco dz ſufficiatur ejus valor, fiet $\frac{z}{dz} = \frac{y}{dy}$; proindeque eadem eſt ſubtangens curvarum ABb, ACc.

PROBL. XI.

55. Sit curva quævis ACc, cujus tangens ad quodlibet punctum innoteſcat; ducatur ejus ordinata BC, quæ protrahatur verſus E ita ut cubus CE ſit ubique æqualis facto ex reſpondente arcu AC in quadratum datæ rectæ, & hoc ſemper fiat, quæritur tangens curvæ AEe, quam puncta ſic inventa determinant. *Fig.* 15.

Ducatur recta be infinite proxima ipſi BE, & ex punctis, C, E ducantur normales CD, EF ad ordinatam be. Sit arcus $AC = x$, $BC = y$, & data recta ſit $= a$, erit $CE = \sqrt[3]{aax}$, proindeque $BE = y \pm \sqrt[3]{aax}$, & $eF = dy \pm \frac{aadx}{3\sqrt[3]{a^4xx}}$. Sit præterea ſubtangens $TB = s$, & tangens $TC = t$. Jam vero ob ſimilit. triang. CTB, cCD eſt $CT : TB = cC : CD$, $t : s = dx : \frac{sdx}{t} = CD = EF$. Præterea ob

ſimilit.

ſimilit. triang. eFE, EBt eſt $eF : FE = EB : Bt$,

$$dy + \frac{aadx}{3\sqrt[3]{a^4xx}} : \frac{sdx}{t} = y + \sqrt[3]{aax} :$$

$$\frac{3sydx\sqrt[3]{a^4xx} + 3aasxdx}{3tdy\sqrt[3]{a^4xx} + aatdx} = Bt.$$ Sed cum ſit TB: BC = CD : DC, $s : y = \frac{sdx}{t} : dy$, erit $dy = \frac{ydx}{t}$: quare ſi in valore ipſius Bt loco dy ſubſtituatur $\frac{ydx}{t}$, erit $Bt = \frac{3sy\sqrt[3]{axx} + 3asx.}{3y\sqrt[3]{axx} + at}$

SCHOL.

56. Doctiſſimus TSCHIRNAUSIUS in ea fuit ſententia duci poſſe tangentem ad hujuſmodi curvam, ut & ad omnes curvas mechanicas ſine præſidio calculi differentialis, vel, ut ait ipſe, nulla poſita magnitudine indefinite parva. Quare non abs re erit hoc loco animadvertere Virum Cl. in hujuſmodi tangente determinanda ad quantitates infinite parvas nil tale cogitantem deſcendiſſe. Ita enim probl. conficit Vir Sapientiſſimus.

„ Ducantur rectæ EM, CH, quæ alicubi concurrent
„ in L, quæque ſecabunt curvas AEM, ACH in
„ punctis M, H, E, C, quæ ſunt ad duas communes
„ ordinatas GM, BE. Jam ob ſimilitud. triang.
„ MHL, MKE eſt MH : HL = MK : KE = HC.
„ Sed ubi [ob motum ipſius GM verſus BE] rectæ
„ HL, LM non amplius ſecabunt curvas ACH,
„ AEM, tunc eædem tangent curvas ipſas in E, &
„ C, & arcus CH fiet = chordæ CH. Hactenus
„ Vir Cl. Patet

Patet autem hic tacite ſupponi quantitates infinite parvas, in quibus arcus cum chorda convenire poſſit.

PROBL. XII.

57. Sit curva ABCD, cujus tangens BT; extendantur ejus ordinatæ KB ultra perimetrum, & ponatur BG = portioni axis DK, vel coordinatæ BE, quæritur tangens curvæ AGH.

Fig. 16.

Sit AE $= x$, EB $= y$, AD $= a$; erit DE = KB $= a - x$, & ob BG = BE $= y$ eſt KG $= a - x + y$, quare ejus infiniteſima portio mL $= dy - dx$ ad LG = BN $= dy$, ut ſubtangens TF ad FG $= y$; proindeque TF $= \frac{ydy - ydx}{dy} = y - \frac{ydx}{dy}$; ergo quoniam EF = BG $= y$, & ET $= \frac{ydx}{dy}$, eſt enim ſubtangens curvæ ABC, erit TF ſubtangens curvæ AGH = EF — ET = TF; ergo ex eodem puncto T profluunt tangentes utriuſque curvæ.

Aliter. Tangat BT priorem curvam in B, juncta GT erit tangens quæſita; Non enim, ſed alia Gt tangat, ponaturque GM parallela ipſi BT, erit propterea BP : BN = TB : BE. Sed ipſi BP eſt = GO, ipſi BN = mO, & ipſi BT = GM, ergo GO : Om = GM : BE = GM : Mt, ergo BE = BG = Mt. Sed MT = BG in parallelogrammo TBGM, ergo puncta t, T coincidunt, ideoque juncta TG erit tangens.

PROBL. XIII.

58. Data ſit curva KFD, cujus nota ſit tangens in quolibet puncto. Ex A ducantur utcumque rami AF, AD, in quibus reſecentur portiones AB, AC æquales reſpective arcubus KF, KD, quæritur tangens curvæ tranſeuntis per omnia puncta tali modo determinata.

Fig. 17.

Ex puncto A ducantur infinite proximæ AF, AD, & centro A intervallis AB, AF deſcribantur

arculi

arculi BG, FE, & in puncto A erigatur AT normalis ad AF, quæ occurret tangenti datæ curvæ in puncto T. Sit arcus $KF = x$, $AF = y$, erit $DE = dy$, $CG = dx$, $AT = s$, $TF = t$, & quæſita tangens ponatur BH. Cum autem ſimilia ſint triang. FAT, DEF, erit $FA : AT = DE : EF$, $y : s = dy : \frac{sdy}{y} = EF$. Præterea ob ſimilit. triang. FAE, DEF erit $FA : AB = FE : BG$, $y : x = \frac{sdy}{y} : \frac{sxdx}{yy} = BG$, & ob ſimlit. triang. BAH, CGB eſt $CG : GB = BA : AH$, $dx : \frac{sxdy}{yy} = x : \frac{sxxdy}{yydx} = AH$. Denique ob ſimilit. triang. TFA, FDE eſt $TF : FA = FD : DE$, $t : y = dx : \frac{ydx}{t} = DE$. Sed $DE = dy$, ergo $dy = \frac{ydx}{t}$; quare ſi in $\frac{sxxdy}{yydx}$ loco dy ſufficiatur $\frac{ydx}{t}$, erit $\frac{sxxdy}{yydx}, = \frac{sxxydx}{tyydx} = \frac{sxx}{ty} = AH$ ſubtangenti.

COROLL.

59. Si curva KFD ſit circulus, cujus centrum A, quoniam ejus tangens FT ad infinitam diſtantiam convenit cum ſubtangente AT, ſiquidem illi parallela eſt, cum utraque ſit ramo AF perpendicularis, fiet $s = t$, & utraque infinita, quare formula ſubtangentis fiet $= \frac{xx}{y}$.

PROBL. XIV.

60. Sit propoſita curva ABC, cujus tangens ad quodlibet punctum innoteſcat, & ex puncto P ſumto in ejus plano ducantur utcumque ad curvam rectæ *Fig. 18.*

PB,

P B, P D, quæ ita protrahantur in F, G, ut ſit B F $=$ A B, D G $=$ A D, & ſic ſemper, conſtruetur alia curva, cujus quæritur tangens.

Sit recta quævis K N parallela ipſi P F, ad quam ex punctis B, F ducantur normales B L, F K, ductaque tangente B N ad curvam propoſitam, ponatur F M quæſita tangens curvæ genitæ A F G. Sint autem P F, P G infinite proximæ, & centro P deſcriptis arculis B E, F H, ſit B L $= a =$ F K, B N $= t$, L N $= s$, BP $=$ y, A B $=$ BF $=$ x, eritque P F $=$ y $+$ x, B D $= d$x, D E $= d$y, G H $= d$y $+$ dx. Similia ſunt triang. L B N, D B E; Cum enim ang. B E P ſit rectus, & ang. B P E infinite parvus, ang. P BE non differet a recto; ſed ang. P B L eſt rectus; nam ducta fuit B L normalis rectæ K N parallelæ ipſi F P; ergo ad datam rectam PB, atque ad ejus punctum B duæ rectæ BE, BL non ad eaſdem partes ductæ eos, qui ſunt deinceps, angulos duobus rectis æquales faciunt, adeoque in directum ſunt inter ſe ipſæ rectæ lineæ, quare ang. LBN $=$ DBE, & ang. ad E, & L ſunt recti quare MB : BL $=$ BD : BE, t : $a = d$x : $\frac{a\,d\mathrm{x}}{t} =$ BE. Similia quoque ſunt triang. BPE, FPH, ergo BP : PF $=$ BE : FH, y : y $+$ x $=$ $\frac{a\,d\mathrm{x}}{t} : \frac{a\mathrm{y}\,d\mathrm{x} + a\mathrm{x}\,d\mathrm{x}}{t\mathrm{y}} =$ FH. Denique cum ſimilia pariter ſint triang. FHG, FKM, erit FH : HG $=$ FK : KM, $\frac{a\mathrm{y}\,d\mathrm{x} + a\mathrm{x}\,d\mathrm{x}}{t\mathrm{y}} : d\mathrm{y} + d\mathrm{x} = a : \frac{(a\,d\mathrm{y} + a\,d\mathrm{x})t\mathrm{y}}{a\mathrm{y}\,d\mathrm{x} + a\mathrm{x}\,d\mathrm{x}}$ $=$ KM. Sed eſt BN : LN $=$ BD : DE, t : $s = d$x : $\frac{s\,d\mathrm{x}}{t} =$ DE $= d$y, ergo $\frac{s\,d\mathrm{x}}{t} = d$y, & neceſſariis inſtitutis ſubſtitutionibus fiet KM $= \frac{s\mathrm{y} + t\mathrm{y}}{\mathrm{x} + \mathrm{y}}$.

Simili methodo duci poſſunt tangentes ad omnes curvas, ut patet.

PROBL.

PROBL. XV.

61. Sit curva quævis A B C, cujus nota ſit tangens in quolibet puncto B, & alia curva ejus proprietatis, ut ordinatæ H D ſint ſemper mediæ proportionales inter H B ordinatam prioris curvæ, & abſciſſam H A, quæritur tangens t D. *Fig.* 19.

Sit ordinata L E infinite proxima ipſi HD, ſitque HB $= y$, HA $= x$, HD $= z$, erit BG $=$ HL $=$ DF $= dx$, CG $= dy$, EF $= dz$. Similia ſunt triang. EFD, DHt, ergo EF : FD $=$ DH : Ht, $dz : dx = z : \frac{zdx}{dz} = Ht$. Cum autem æquatio curvæ ſit $zz = xy$, erit $2zdz = xdy + ydx$, & $dz = \frac{xdy + ydx}{2z}$, ergo

$$Ht = \frac{zdx}{zd} = \frac{2zzdx}{xdy + ydx}.$$

COROLL. I.

62. Hinc ſi data curva A B C fuerit parabola æquationis $yy = ax$, erit $2ydy = adx$, & $dy = \frac{adx}{2y}$; quare ſubſtituendo erit $\frac{2zzdx}{xdy + ydx} = \frac{4yzzdx}{axdx + 2ydx}$ $= \frac{4xyy}{ax + 2yy} = \frac{4}{3}x = Ht$.

COROLL. II.

63. Si curva ABC fuerit ſemicirculus, erit $Ht = \frac{2zzdx}{xdy + ydx} = \frac{4ax - 4xx}{3a - 4x}$. Et ſic deinceps.

PROBL. XVI.

64. Sint omnia ut ſupra; ſed ordinata normalis B D ſit ſemper media proportionalis inter AH, & HE; quæritur tangens curvæ ADE.

Ducatur recta D e parallela rectæ T B b tangenti curvam A B C. Sit B D $= y$, A H $= x$, H L $= a - x$, B H $= z$, erit K $e = dy$, G $b = dz$, B G $= dx$. Eſt quidem K e : e D $=$ D B : B c, $dy : V(dx^2 + dz^2) = y : \frac{y}{dy} V(dx^2 + dz^2)$. Atqui curvæ æquatio eſt $y^2 = ax - x^2$, quare $2ydy = adx - 2xdx$, & $\frac{y}{dy} = \frac{2yy}{adx - 2xdx} = \frac{2ax - 2xx}{adx - 2xdx}$; proindeque B $c = \frac{(2ax - 2xx)}{adx - 2xdx} V(dx^2 + dz^2)$.

COROLL.

65. Si curva A B C ſit parabola Apolloniana parametri a, erit $z = Vax$, & $dz = \frac{adx}{2Vax}$, $dz^2 = \frac{adx^2}{4x}$; ergo $V(dx^2 + dz^2) = dx\sqrt{\frac{(4x + a)}{4x}}$, & $\frac{(2ax - 2xx)V(dx^2 + dz^2)}{adx - 2xdx} = $ B $c = \frac{(2ax - 2xx)}{a - 2x}\sqrt{\frac{(4x + a)}{4x}} = \frac{(a - x)}{a - 2x}V(4xx + ax)$.

PROBL. XVII.

66. Sint datæ curvæ A Q C, A N R, quarum communis axis ſit recta A P, earumque tangentes N S, Q T innoteſcant, curvas autem datas ſecet ordinata N Q P, quæ protrahatur in M, ut ſit P Q : P N $=$ P N : P M, quæritur tangens curvæ A M.

Fig. 20

Ponatur quæſita tangens M V, & ordinetur R C p G infinite proxima ordinatæ N Q P M, ductiſque N n, Q q, M m

Mm axi AB parallelis ponatur $PT = s$, $PS = r$, $PQ = y$, $PN = z$, erit $PM = \frac{zz}{y}$; $qC = dy$, $nR = dz$, $Gm = \frac{2yzdz - zzdy}{yy}$, $Pp = dx$. Eſt autem ob ſimilitud. triangul. QPT, CqQ, $PQ : PT = Cq : qQ$, $y : s = dy : \frac{sdy}{y} = qQ = Pp = Mm = dx$, hinc $dy = \frac{ydx}{s}$. Præterea ob ſimilit. triang. NPS, RnN eſt $NP : PS = Rn : nN$, $z : r = dz : \frac{rdz}{z} = nN = Pp = dx$, quare $dz = \frac{zdx}{r}$. Denique ob ſimilit. triang. GmM, MPV eſt $Gm : mM = MP : PV$, $\frac{2yzdz - zzdy}{yy} : dx = \frac{zz}{y} : \frac{yzdx}{2ydz - zdy} = PV$; & ſi loco dz, & dy ſubſtituantur eorundem valores nuper inventi, erit $\frac{yzdx}{2ydz - zdy} = \frac{rs}{2s - r}PV$.

Quod ſi iiſdem ut ſupra poſitis problema Hoſpitaliano more enuncietur, nimirum quod relatio adplicatarum PQ, PN, PM quavis æquatione exprimatur, retentis iiſdem denominationibus, poſitaque $PM = v$, adeoque $Gm = dv$, quoniam $\frac{sdy}{y} = Qq = \frac{rdx}{z} = Nn$, erit $dz = \frac{szdy}{ry}$. Præterea eſt $Gm : mM = MP : PV$, $dv : \frac{sdy}{y} = v : \frac{svdy}{ydv}$, qui eſt valor quæſitæ ſubtangentis.

Quod ſi in differentia datæ æquationis fiant neceſſariæ ſubſtitutiones, evaneſcent quantitates infinite

 parvæ,

Digitized by Google

parvæ, adeoque ſubtangentis valor per quantitates finitas exprimetur.

Sit ex. gr. $zz = vy$, ergo $2zdz = vdy + ydv$, & loco dz ſuffecto ejus valore $\frac{szdy}{ry}$, erit $\frac{2szzdy}{ry} = vdy + ydv$; hinc $\frac{2szzdy - rvydy}{ryy} = dv$, adeoque $\frac{svdy}{ydv} = \frac{rsvy}{2szz - rvy} = \frac{rs}{2s - r}$, ſi loco zz ſufficiatur vy.

PROBL. XVIII.

67. Sint omnia ut in probl. præcedente, & fiat ordinata $PM = AQ + AN$, quæritur tangens curvæ tranſeuntis per omnia puncta tali ratione determinata.

Fiant omnia ut ſupra, & dicantur $NS = t$, $QT = p$, $AN = z$, $AQ = v$; erit ob ſimil. triang. QTP, CQq, $QT : TP = CQ : Qq$, $p : s = dv : \frac{sdv}{p} = dx$, hinc $dv = \frac{pdx}{s}$, itemque ob ſimilit. triang. NSP, RNn erit $NS : SP = RN : Nn$, $t : r = dz : \frac{rdz}{t} = dx$, ergo $dz = tdx$. Denique ob ſimilit. triang. GmM, MPV eſt $Gm : mM = MP : PV$, $dz + dv : dx = z + v : \frac{zdx + vdx}{dz + dv} = \frac{rsz + rsv}{ts + pr} = PV$.

PROBL. XIX.

68. Sit circulus CEF, & alius duplæ diametri ABF, & ducta chorda AB, demiſſaque ex B recta BD normali in diametrum AF occurrente alteri circumferentiæ in E, fiat $AK =$ chordæ EC, & ſic ſemper, quæritur tangens curvæ, quæ tranſeat per omnia puncta ſic inventa. *Fig. 21.*

Hujuſmodi

Hujuſmodi Curva eadem eſt cum Lemniſcata Cl. JACOBI BERNOULLI.

Sint infinite proximæ AB, AH, & in puncto A erecta ad AB normali AT, quæ occurrat in T rectæ BT tangenti circulum in puncto B, centro A intervallis AK, AB deſcribantur arculi infinite parvi KL, BG, & ſit $AT = s$, $CD = x$, $AC = a$, erit $AD = a + x$, $DF = a - x$, & $CE = AK = \sqrt{ax}$, proindeque $ML = \frac{adx}{2\sqrt{ax}}$. Sed $AB = \sqrt{[2aa + 2ax]}$, ergo $HG = \frac{adx}{\sqrt{[2aa + 2ax]}}$, quæſita vero tangens ponatur KS. Jam vero ob ſimilit. triang. BAT, HGB eſt $BA : TA = HG : GB$, $\sqrt{(2aa + 2ax)} : s = \frac{adx}{\sqrt{(2aa + 2ax)}} : \frac{asdx}{2aa + 2ax}$, & ob ſimilit. triang. BAG, KAL eſt $BA : KA = BG : KL$, $\sqrt{[2aa + 2ax]} : \sqrt{ax} = \frac{asdx}{2aa + 2ax} : \frac{asdx\sqrt{ax}}{(2aa + 2ax)\sqrt{(2aa + 2ax)}}$. Denique ob ſimilit. triang. MLK, KAS eſt $ML : LK = KA : AS$, $\frac{adx}{2\sqrt{ax}} : \frac{asdx\sqrt{ax}}{[2aa + 2ax]\sqrt{[2aa + 2ax]}} = \sqrt{ax} : \frac{sx\sqrt{ax}}{(a + x)\sqrt{(2aa + 2ax)}} = AS$.

PROBL. XX.

69. Invenire ſubtangentem ad determinatum curvæ punctum.

Sumatur differentia datæ æquationis, ex eaque inveniatur valor ſubtangentis $\frac{ydx}{dy}$, & loco x, & y in valore ſubtangentis poſitis valoribus, quos habent in dato puncto, invenietur quæſita ſubtangens.

CO-

COROLL.

70. Quoniam igitur in circulo $\frac{y\,dx}{dy} = \frac{2yy}{a-2x}$, ſi quæratur ſubtangens ad curvæ punctum, cui reſpondet abſciſſa $x = \frac{1}{2}a$, erit $y = \frac{1}{2}a$, hinc $\frac{2yy}{a-2x} = \frac{\frac{1}{2}aa}{0}$, nempe ſubtangens erit infinita, & tangens normalis radio.

Aliter problema demonſtratur.

Quoniam valor ſubtangentis in qualibet curva eſt $= \frac{x\,dy}{dx}$ [§. 49.] si fiant omnia quæ ſupra, eruetur quæſita ſubtangens.

COROLL.

71. Quoniam igitur in circulo eſt $\frac{x\,dy}{dx} = y - \frac{ax}{2y}$ [§. 51.] ſi quæratur ſubtangens ad curvæ punctum, cui reſpondet abſciſſa $x = \frac{1}{2}a$, erit $y - \frac{ax}{2y} = \frac{1}{2}a - \frac{\frac{1}{2}aa}{a} = 0$, nempe nulla eſt ſubtangens, adeoque tangens coincidit cum recta, quæ normalis eſt radio, quæque proinde circulum tangit in puncto extremo radii.

SCHOL. I.

72. Si propoſita curva plures habuerit ramos inter ſe concurrentes, & tangens in puncto concurſus fuerit eruenda, ſive punctum illud ſit punctum contactus, ſive interſectionis, ſuffectis loco x, & y valoribus, quos habent in dato puncto, termini omnes in numeratore, *Fig. 22.* ac denominatore fractionis evaneſcent, & ſubtangens fiet $= \frac{0}{0}$, nempe infinita. Ita ſi in curva æquationis

nis $y^4 - 8by^3 - 12bxyy + 16bbyy + 48bbxy + 4bbxx - 64b^3x = 0$ a Cl. SAURINO propoſitæ in *Ac. reg.* 1716. quæratur ſubtangens in puncto H concurſus crurum AC, DB, cui reſpondet $x = y = 2b$, quoniam in eadem curva punctum A eſt punctum originis ipſarum y, & recta AD earundem axis, proindeque formula ſubtangentium in hoc axe ſumtarum $= \frac{xdy}{dx}$, erit hæc ſubtangens $=$

$$\frac{3bxyy - 12bbxy - 2bbxx + 16b^3x}{y^3 - 6byy - 6bxy + 8bby + 12bbx} = \frac{0}{0} = \infty,$$

ſi $2b$ loco x, & y, & loco x^2, y^2, y^3 eorundem reſpectivi valores ſubſtituantur. Non hinc tamen inferre licet infinitam eſſe in puncto illo ſubtangentem, ſed ſubtangentem aliquam finitam, imo duplicem eidem puncto reſpondere; quæ ut veſtigetur, ſumantur differentiæ ſecundæ propoſitæ æquationis pro conſtantibus adſumendo dx, & dy, & erit (ut inferius conſtabit) $3y^3dyy - 12bydyy - 6bxdyy - 12bydydx + 8bbdy^2 + 24bbdxdy + 2bbdx^2 = 0$. Loco x, & y ſufficiatur eorum communis valor $2b$ relate ad punctum H, & erit $-16bbdy^2 = 2bbdx^2$; vel $2bbdx^2 = 16bbdy^2$, proindeque $dx = \pm dy\sqrt{8}$. Denique in valore ſubtangentis generali $\frac{xdy}{dx}$ poſitis loco x, & dx valoribus $2b$, & $\pm dy\sqrt{8}$, fiet eadem ſubtangens $= \pm \frac{b}{\sqrt{2}}$. Igitur ſubtangens in puncto H, vel potius duæ ſubtangentes puncto H reſpondentes ſunt $= \pm \frac{b}{\sqrt{2}}$, quarum altera eſt poſitiva, altera vero negativa, & æqualis poſitivæ. Ra-

Ratio methodi demonſtratur.

Differentia propoſitæ æquationis numeris omnibus abſoluta eſt.

$$
\begin{array}{lllll}
\quad 1 & \quad 2 & \quad 3 & \quad 4 & \\
+4y^3dy & +6yydy^2 & +4ydy^3 & +dy^4 & \\
-24byydy & -24bydy^2 & -12bdxdy^2 & & \\
-24bxydy & -12bxdy^2 & -8bdy^3 & & \\
-12byydx & -24bydxdy & & & =0 \\
+32bbydy & +16bbdy^2 & & & \\
+48bbydx & +48bbdxdy & & & \\
+48bbxdy & +4bbdx^2 & & & \\
+8bbxdx & & & & \\
-64b^3dx & & & &
\end{array}
$$

Quoniam vero ob varias dimenſiones terminorum columnas haſce conſtituentium, eædem columnæ varios exhibent ordines quantitatum non quidem abſolute $=0$, ſed infinities parvarum, quarta columna erit infinite parva relate ad tertiam, tertia relate ad ſecundam, ſecunda relate ad primam, adeoque prima columna differentiam propoſitæ quantitatis conſtituet, quæ ſi fuerit abſolute $=0$, non erit profecto $=0$ ſecunda columna, quæ proinde differentiam propoſitæ quantitatis exhibebit, quod idem eſt ac differentias ſecundas ejuſdem æquationis invenire.

Si curva, cujus quæritur tangens tres habuerit ramos ad datum punctum concurrentes tertiæ differentiæ erunt inveniendæ, quartæ, ſi curva quatuor habuerit ramos; & ſic deinceps. Sed de his plura acutiſſimus Geometra VINCENTIUS RICCATUS *in Comment. Acad. Bonon.* T. 2. p. 3., & MARIA CAJETANA AGNESI Fœmina longe Clariſſima in ſuis inſtitutionibus Algebraicis. Plura etiam habent

Mathematicus

Mathematicus plane ſummus JACOBUS BERNOULLIUS, & SAURINUS loco ſupra citato.

SCHOL. II.

73. In ſuperiore Scholio innuimus $\frac{0}{0}$ eſſe = oo. Id nunc nobis eſt oſtendendum. Sit itaque AC radius quadrantis circuli ADB, cujus ordinata DE, & ſecans ADG, quæ in puncto G occurrat rectæ CB productæ. Sit AC = a, DE = y, erit propterea AE = $a - V(aa - yy)$. Eſt autem AE : ED = AC : CG = DF : FG, $a - V[aa - yy] : y = V[aa - yy] : \frac{yV(aa - yy)}{a - V(aa - yy)}$ = FG, ergo cum y ſit = o, recta AG erit parallela rectæ CG; quare tunc AG fiet infinita; in hoc vero caſu inventa formula evadet $\frac{0}{0}$, ergo $\frac{0}{0}$ = oo = infinito. *Fig. 23.*

LEMM. I.

74. Sint duo triang. CBA, DEF, & latus DE ſit = lateri CB, ang. vero FDE ſit > ang. ACB, dico latus EF > latere BA. *Fig. 24.*

Fiat enim ang. EDG = ang. ACB, erit GE = AB; ſed latus EF > latere EG; nam recta DG ſecat ang. EDF, ergo latus EF > latere AB.

LEMM. II.

75. Si curva AGB concavitatem axi obvertat, ejuſque ordinatæ ſemper creſcant, differentiæ ordinatarum ſemper magis, ac magis decreſcunt.

Ponimus hic lineam abſciſſarum ſectam eſſe in portiunculas numero infinitas AF, Ff, fb &c. quæ ſunt differentiæ infinite parvæ abſciſſarum AF, Af, Ab &c. adeoque differentias haſce omnes ſive dx eſſe conſtantes; eas autem tales ponimus, ut habeatur veluti punctum fixum in relationibus diverſarum coordinatarum $\frac{gp}{pG}$, $\frac{Kr}{rg}$ conſiderandis. *Fig. 25.*

Digitized by Google

Sint ordinatæ FG, fg, bK, & concipiantur ductæ tangentes GT, gt. Eſt ang. GTF = ang. gGp, & ang. Kgr = ang. gtf. Sed ang. GTF > ang. gtf, ergo ang. gGp > ang. Kgr. Atqui latus Gp eſt = lateri gr, ergo latus gp > latere Kr.

COROLL. I.

76. Unde conſtat hujuſmodi curvas ſemper minus, ac minus ſupra axem elevari, adeoque latus curvæ infinite parvum, nempe DB, vel ejus tangentem BL eſſe axi AH parallelam, & dy fieri infinite parvam relate ad dx, & quidem juxta omnes infiniteſimi gradus, quin unquam ad zero abſolutum pertingat; Hoc enim, ut patet, contingit tantummodo in lineis inter ſe abſolute parallelis.

COROLL. II.

77. Ergo quo minor erit valor ipſius dy, eo magis latuſculum DB, vel ejus tangens BL fiet axi AH parallela; linea quippe, quæ alteri pro parallela habetur, eò magis ipſi parallela eſſe poteſt juxta quemlibet gradum. Et quidem in triang. ABC ſi baſis BC ponatur infiniteſima primi ordinis latera AB, AC cenſeri poterunt pro parallelis ob ang. A infinite parvum,

Fig. 26. quo in caſu ipſa EC, ſive dy fiet $= \frac{1}{oo^2}$. Quod ſi præter baſim infiniteſimam ponamus verticem A infinite diſtare a baſi BC, latera ipſa AB, AC fient inter ſe magis parallela, & quidem tum ratione ang. infiniteſimi A, tum ratione infinitæ diſtantiæ ejuſdem ang. a baſi, & ipſa EC = dy fiet $= \frac{1}{oo^3}$. Jam vero ſi præter infinitam diſtantiam verticis A a baſi ponamus hanc eſſe infiniteſimam ſecundi ordinis, adhuc etiam latera AB, AC fient inter ſe magis parallela, & ipſa EC = dy erit $= \frac{1}{oo^4}$; & ita porro.

PRO-

Digitized by Google

PROBL. XXII.

78. Invenire ſubtangentem in curvis, quarum latus infinite parvum protracta in infinitum curva ſit axi parallelum.

Sumatur curvæ æquatio, & adſignetur ordinatæ y conveniens valor infinitus, mox eruatur ex data æquatione valor ipſius dy, & ſolutum erit problema.

COROLL. I.

79. Quoniam igitur in parabola Apolloniana eſt $y = oo^{\frac{1}{2}}$, quippe, ut ait Cl. FONTENELLIUS, & ratio ipſa evincit, de duabus quantitatibus, quæ uno eodemque tempore fiunt infinitæ, vel infinite parvæ, judicandum eſt ex peculiari charactere, quem iiſdem tribuit neceſſitas relationis inter ipſas intercedentis, & $dx = \frac{1}{oo}$, erit $dy = \frac{dx}{2\,oo^{\frac{1}{2}}} = \frac{1}{2\,oo^{\frac{3}{2}}}$, & $\frac{y\,dx}{dy} =$

$$\frac{oo^{\frac{1}{2}} \cdot \frac{1}{oo}}{\frac{1}{2\,oo^{\frac{3}{2}}}} = 2\,oo.$$

COROLL. II.

80. Eadem ratione in prima parabola cubica $y^3 = x$, cum ſit $y = oo^{\frac{1}{3}}$, & $dx = \frac{1}{oo}$ erit $dy = \frac{dx}{3\,oo^{\frac{2}{3}}} = \frac{1}{3\,oo^{\frac{5}{3}}}$, & $\frac{y\,dx}{dy} = 3\,oo$.

COROLL. III.

81. Et generatim in primis parabolis cujuſvis ordinis cum ſit $y^m = x$, erit $y = oo^{\frac{1}{m}}$, Eſt autem $dx = my^{m-1}dy$, eademque $= \frac{1}{oo}$, ergo $dy = \frac{1}{m\,oo^{\frac{2m-1}{m}}}$

& eſt $ydx = oo^{\frac{1}{m}} . \frac{1}{oo} = \frac{oo^{\frac{1}{m}}}{oo} = oo^{\frac{1-1}{m}} = oo^{\frac{1-m}{m}}$, quare $\frac{ydx}{dy} = m\,oo$.

COROLL. IV.

82. Patet igitur, quo major eſt dimenſio parabolæ, eo majorem fieri $\frac{2m-1}{m}$, adeoque ſemper decreſcere valorem ipſius dy, ac proinde parabolam majoris dimenſionis ad ſui extremitatem fieri ſemper magis axi parallelam.

COROLL. V.

83. In parabola ſecunda ſecundi ordinis $y^3 = x^2$ eſt $y = oo^{\frac{2}{3}}$, ergo $dy = \frac{2}{3\,oo^{\frac{4}{3}}}$, & $ydx = oo^{-\frac{1}{3}}$, proindeque $\frac{ydx}{dy} = \frac{oo^{-\frac{1}{3}} . 3\,oo^{\frac{4}{3}}}{2} = \frac{3}{2}\,oo$.

COROLL. VI.

84. Generatim vero in omnibus parabolis ſecundis cujuſvis ordinis cum ſit $y^m = x^{m-1}$ erit $y = oo^{\frac{m-1}{m}}$,

& dy

& $dy = \frac{m-1}{m\,oo\,\frac{m+1}{m}}$, proindeque $\frac{y\,dx}{dy} = \frac{m\,oo}{m-1}$.

COROLL. VII.

85. Quoniam vero, quo major eſt gradus parabolarum ſecundarum, eo fit minor $\frac{m+1}{m}$, adeoque augetur ſemper valor ipſius dy, patet, quo major eſt dimenſio parabolarum ſecundarum, eo minus eaſdem fieri ad ſui extremitatem axi parallelas.

LEMM.

86. Si curva convexitatem axi obvertat, & ordinatæ ſemper creſcant, differentiæ ordinatarum ſemper magis, ac magis creſcent. *Fig.* 27.

Hic pariter ponuntur dx conſtantes.

Demonſtratio hujus lemm. convenit cum demonſtratione lemm. præcedentis.

COROLL. I.

87. Unde colligitur, quod cum in triang. KqB ang. qKB fiat ſemper major, latus BK ſemper minus obliquum evadet lateri Kq cui tandem producta in infinitum curva latus ipſum vel potius ejus tangens cenſeri poterit ut perpendicularis, & dy, erit infinite magna relate ad dx, & quidem juxta omnes infiniti radicalis dimenſiones, ut patet, ſi conſideretur latus Bq tanto fieri majus, quo latus BK fit magis perpendiculare lateri Kq.

COROLL. II.

88. Conſtat igitur curvam tanto magis ad ſui extremitatem fieri normalem axi, quo major eſt valor ipſius dy.

PROBL. XXIII.

89. Invenire ſubtangentem in curvis, quarum latus infinite parvum abeunte in infinitum curva fit axi perpendiculare.

Fiant

Fiant quæ in Superiore problem. præcepimus, & confectum erit problema.

COROLL. I.

90. Quoniam igitur æquatio ad parabolam communem, ſi ejus axis ponatur $AD = x$, ordinata $DM = y$, eſt $y = x^2$, & eſt $y = oo$, erit ergo $x = oo^{\frac{1}{2}}$. Eſt autem $dx = \frac{1}{oo}$, ergo dy erit $= 2\,oo^{\frac{1}{2}} . \frac{1}{oo} = 2\,oo^{-\frac{1}{2}}$;

quare $\frac{y\,dx}{dy} = \frac{1}{2\,oo^{-\frac{1}{2}}} = \frac{oo^{\frac{1}{2}}}{2}$.

COROLL. II.

91. Pariter cum in parabola æquationis $y = x^3$, ſit $x = oo^{\frac{1}{3}}$, erit $dy = 3\,oo^{-\frac{1}{3}}$, & $\frac{y\,dx}{dy} = \frac{oo^{\frac{1}{3}}}{3}$.

COROLL. III.

92. Eademque ratione demonſtrabitur ſubtangentem curvæ æquationis $y = x^4$ eſſe $= \frac{oo^{\frac{1}{4}}}{4}$, atque ita porro.

COROLL. IV.

93. Ex his colligitur quod quo major eſt primarum parabolarum dimenſio, eo major evadit valor ipſius dy, proindeque quo primæ parabolæ majoris ſunt dimenſionis, eo magis ad ſui extremitatem fiunt axi perpendiculares.

CO-

COROLL. V.

94. Quod ſi fuerit æquatio ad ſecundam parabolam cubicam, $y^2 = x^3$ quoniam eſt $x = oo^{\frac{2}{3}}$, & $dy = \frac{3}{2} oo^{-\frac{2}{3}}$, erit $ydx = \frac{2}{3\, oo^{-\frac{2}{3}}}$, & $\frac{ydx}{dy} = \frac{2}{3} oo^{\frac{2}{3}}$.

COROLL. VI.

95. Pariter in curva æquationis $y^3 = x^4$ cum ſit $x = oo^{\frac{3}{4}}$, & $dy = \frac{4}{3} oo^{-\frac{3}{4}}$, erit $\frac{ydx}{dy} = \frac{3}{4} oo^{\frac{3}{4}}$, & ſic deinceps.

COROLL. VII.

96. Hinc patet, quo major eſt gradus parabolarum ſecundarum, eo minus eaſdem ad ſui extremitatem fieri normales axi, ſcilicet ob valorem ipſius dy ſemper minorem.

PROBL. XXIV.

97. Invenire tangentem ad verticem propoſitæ curvæ.

Inveniatur ex æquatione curvæ ratio ipſius $dy : dx$, & problema confectum erit, ut exemplis conſtabit.

COROLL. I.

98. Quoniam in parabola eſt $2y\,dy = dx$, erit $dy : dx = 1 : 2y$, nempe ut ſin. rectus M P ad ſin. complementi P T, ita parameter ad duplum ordinatæ, vel dimidium parametri ad ordinatam, ubicumque punctum M reperiatur; at in vertice A eſt $y = o$, vel potius $= \frac{1}{oo}$, erit igitur $dy : dx = 1 : \frac{1}{oo}$, nempe ratio ipſius $dy : dx$ infinite magna, ſcilicet nullus erit ſinus complementi, adeoque tangens ad verticem normalis Fig. 5.

eſt

Digitized by Google

eſt axi, cui pariter normalis eſt ipſa curva parabolica; Eſt autem $dy = \frac{1}{oo}$, ergo $dx = \frac{1}{oo^2}$.

COROLL. II.

99. In prima parabola cubica eſt $y^3 = x$, & $3yydy = dx$, ergo $dy : dx = 1 : 3yy$; eſt autem $y = \frac{1}{oo}$, ergo $yy = \frac{1}{oo^2}$, proindeque ratio ipſius $dy : dx$ infinita, & quidem ſecundi gradus; tangens igitur ad verticem hujuſce curvæ eſt axi perpendicularis, cui etiam perpendicularis eſt curva ipſa, & $dx = \frac{1}{oo^3}$.

COROLL. III.

100. Quoniam vero in primo Coroll. eſt $dx = \frac{1}{oo^2}$, & in ſecundo eſt $dx = \frac{1}{oo^3}$, patet quo major eſt primarum parabolarum dimenſio, eo magis normales eaſdem fieri ad axem in vertice.

COROLL. IV.

101. Si fuerit æquatio ad ſecundam parabolam cubicam $y^3 = xx$, erit $3yydy = 2xdx$, & $dy : dx = 2x : 3yy$; Eſt vero $yy = \frac{1}{oo^2}$, & $x = \frac{1}{oo^{\frac{3}{2}}}$, ergo ratio $dy : dx$ infinita, & tangens verticis eſt axi normalis; ipſa vero $dx = \frac{3}{2\,oo^{\frac{3}{2}}}$.

COROLL. V.

102. Item ſi detur æquatio ad ſecundam parabolam quadrato-cubicam $y^5 = x^4$, erit $5y^4 dy = 4x^3 dx$, & $dy : dx = 4x^3 : 5y^4$; Eſt autem $5y^4 = \frac{5}{oo^4}$, & $4x^3 = \frac{4}{oo^{\frac{7}{2}}}$, ergo ratio $dy : dx$ infinita, adeoque tangens curvæ ad verticem eſt normalis axi, & $dx = \frac{5}{4\,oo^{\frac{5}{4}}}$; atque ita porro.

COROLL. VI.

103. Patet autem ex valore ipſius dx, quod quo major eſt parabolarum ſecundarum dimenſio, eo minus normales eædem fiunt ad axem in origine.

COROLL. VII.

104. Quod ſi detur æquatio ad parabolam communem, cujus axis AD, ordinata DM, nempe $y = xx$, erit $dy = 2x dx$, & $dy : dx = 2x : 1$; Eſt autem ad verticem $x = 0$, vel $\frac{1}{oo}$, & ad 1 rationem habet infiniteſimam, ergo ratio ipſius $dy : dx$ erit pariter infiniteſima, ergo tangens curvæ ad verticem conveniet cum axe, & rectum ang. conſtituet cum ordinatis ad eundem axem; ipſa vero dy erit $= \frac{2}{oo^2}$; eſt enim $dx = \frac{1}{oo}$.

COROLL. VIII.

105\. Si fuerit æquatio ad primam parabolam cubicam $y = x^3$, cujus ſcilicet axis ſit AD, ordinata DM, ut ſupra, erit $dy = 3xxdx$, & $dy : dx = 3xx : 1$; at in vertice eſt $x = \frac{1}{oo}$, ergo $3x^2 : 1$ rationem habet infiniteſimam ſecundi gradus; eandemque habet $dy : dx$, ergo tangens hujus curvæ ad verticem convenit cum axe, & dy erit $= \frac{3}{oo^3}$; & ſic de reliquis parabolis.

COROLL. IX.

106\. Ex his colligitur, quo major eſt primarum parabolarum dimenſio, quarum axis tranſponatur, eo magis parallelas eaſdem fieri in origine; valor quippe ipſius dy ſemper ſit minor.

COROLL. X.

107\. Si vero fuerit æquatio ad ſecundam parabolam cubicam $yy = x^3$, cujus axis, & ordinata ſint quæ ſupra, erit $2ydy = 3xxdx$, & $dy : dx = 3xx : 2y$; atqui ratio $3x^2 : 2y$ eſt eadem cum ratione $\frac{3}{oo^2} : \frac{2}{oo}$, nempe infiniteſima, ergo eadem eſt ratio ipſius $dy : dx$, ergo tangens curvæ ad verticem convenit cum axe, & dy eſt $= \frac{3}{2\,oo^{\frac{3}{2}}}$.

COROLL. XI.

108\. Denique ſit æquatio ad ſecundam parabolam quarti gradus $y^3 = x^4$, cujus axis tranſponatur, erit $3yydy$

$3yydy = 4x^3 dx$, & $dy : dx = 4x^3 : 3yy$; quare cum ratio ipſius $dy : dx$ ſit infiniteſima, tangens hujus curvæ ad verticem congruet cum axe, & dy erit $= \frac{4}{3\,oo^{\frac{4}{3}}}$.

COROLL. XII.

109. Hinc conſequitur ſecundas parabolas tanto minus parallelas fieri in origine, quo major eſt earundem dimenſio; valor quippe ipſius dy ſemper major evadit.

PROBL. XXIV.

110. Invenire formulam tangentis generalem in curva quavis algebraica in ang. coordinatarum recto.

Sint omnia, ut in problem. 5. Et quoniam $Rr = Rm$ (§. 9. 11.) erit $Rm : mM = PM : MT$, nempe $dy : V(dx^2 + dy^2) = y : \frac{y}{dy} V(dx^2 + dy^2)$, quæ eſt formula generalis ad tangentem.

COROLL. I.

111. In parabola igitur tangens erit $= (V4xx + ax)$. Cum enim ſit $\frac{2ydy}{a} = dx$, adeoque $\frac{4yydy^2}{aa} = dx^2$, ſi loco dx^2 ſufficiatur ejus valor in formula nuper inventa, migrabit eadem in $V \frac{[4y^4 + yy]}{aa} = V(4xx + ax)$.

COROLL. II.

112. In circulo quoniam eſt $2ydy = adx - 2xdx$, erit $dx^2 = \frac{4yydy^2}{aa - 4ax + 4xx}$ quare ſubſtitu-

 tione

tione facta erit $\frac{y}{dy} V[dx^2 + dy^2] =$

$$V\frac{[4aaxx - 8ax^3 + 4x^4 + ax - xx]}{aa - 4ax + 4xx}.$$

COROLL. III.

113. Quod ſi detur æquatio $a^m x^n = y^{m+n}$ ad infinitas parabolas, erit $dx = \frac{(m+n)y^{m+n-1}dy}{na^m x^{n-1}}$;

quare $dx^2 = \frac{[m+n]^2 y^{2m+2n-2} dy^2}{n^2 a^{2m} x^{2n-2}}$. Hinc ſi

in formula tangentis generali loco dx^2 ſufficiatur ejus

valor, fiet eadem $= V[\frac{(m+n)^2}{n^2} x^2 + a^{\frac{2m}{m+n}} x^{\frac{2n}{m+n}}]$

Si fuerit $m = 1$, $n = 1$, parabola fiet Apolloniana, & tangens $= V(4xx + ax)$, ut ſupra.

PROBL. XXV.

114. Invenire formulam tangentis generalem in ang. coordinatarum obtuſo, vel acuto.

Sint infinite proximæ GS, OV, quæ cum axe AC ang. obtuſum, vel acutum contineant. Ducatur GH parallela eidem axi AC; Et quoniam in triang. GHO ang. GHO notus eſt, utpote = ang. TSG, ſi ex punctis O, G ducantur GF, OC normales rectæ TC, & producatur, ſi opus fuerit, recta GH uſque ad P, notus erit ang. OHP, ergo cum ſit rectus ang. HPO, notus erit & ang. HOP, proindeque data erit ratio ipſius OH : OP, & ſit ratio

Fig. 28.

$m : n$,

$m : n$, erit igitur $m : n = dy : \frac{n\,dy}{m} =$ OP. Inſuper data erit ratio ipſius OH : HP, ſitque eadem ac ratio $m : p$, erit ergo $m : p = dy : \frac{p\,dy}{m} =$ HP, quare $GP = dx + \frac{p\,dy}{m}$ [num. 1.] & $= dx - \frac{p\,dy}{m}$ [num. 2.]. Hinc $GO^2 = GP^2 + OP^2 = \frac{m^2\,dx^2 \pm 2mp\,dy\,dx + p^2\,dy^2 + n^2\,dy^2}{m^2}$. Sed $HO^2 = HP^2 + PO^2$, & HO exprimitur per m, OP per n, PH per p, ergo $m^2 = n^2 + p^2$, & $m^2\,dy^2 = n^2\,dy^2 + p^2\,dy^2$; igitur in valore ipſius GO^2 loco $n^2\,dy^2 + p^2\,dy^2$ ſuffecto valore m^2, erit $GO^2 = \frac{m^2\,dx^2 \pm 2mp\,dy\,dx + m^2\,dy^2}{m^2}$, & $GO = \sqrt{\frac{(m\,dx^2 \pm 2p\,dy\,dx + m\,dy^2)}{m}}$. Sed triang. GOH, TGS ſunt ſimilia, ergo OH : OG $=$ SG : GT, $dy : \sqrt{\frac{(m\,dx^2 \pm 2p\,dy\,dx + m\,dy^2)}{m}} = y : \frac{y}{dy} \sqrt{\frac{(m\,dx^2 + 2p\,dy\,dx + m\,dy^2)}{m}}$ quæ eſt formula tangentis GT.

PROBL.

PROBL. XXVI.

115. Invenire formulam generalem ſubnormalis in ang. coordinatarum recto.

Sint omnia ut in probl. 5. Et quoniam triang. MRr, PNM ſunt ſimil. erit $RM : Rr = PM : PN$, $dx : dy = y : \frac{ydy}{dx} =$ ſubnormali PN. Hinc ſi ex data curvæ æquatione valor ipſius dy, vel dx eruatur, & in formula ſubnormalis loco ejuſdem dy, vel dx ſufficiatur, ſubnormalis valor in quantitatibus finitis emerget.

Idem ſubnormalis valor eruetur, ſi fiat ſubtangens $TP : PM = PM : PN$, $\frac{ydx}{dy} : y = y : \frac{ydy}{dx}$. Quod ſi loco $\frac{ydx}{dy}$ ejus valor jam inventus ſubſtituatur, ſubnormalis in quavis curva determinabitur.

COROLL. I.

116. Cum itaque in parabola ſit $yy = ax$, & $2ydy = adx$, ſive $\frac{2ydy}{a} = dx$, erit $\frac{ydy}{dx} = \frac{1}{2}a$. Præterea quoniam in eadem parabola eſt $\frac{ydx}{dy} = 2x$, erit $2x : y = y : \frac{yy}{2x} = \frac{ax}{2x} = \frac{1}{2}a$.

COROLL. II.

117. Si fuerit æquatio ad infinitas parabolas $y^{m+n} = a^m x^n$, erit $\frac{ydy}{dx} = \frac{n a^m x^{n-1}}{(m+n) y^{m+n-2}}$ ſi $m = 1$, $n = 1$, erit $\frac{n a^m x^{n-1}}{(m+n) y^{m+n-2}} = \frac{1}{2}a$. Et ſic de ceteris.

CO-

COROLL. III.

118. Quoniam in circulo eſt $2ydy = adx - 2xdx$, erit $\frac{ydy}{dx} = \frac{a - 2x}{2} = \frac{1}{2}a - x$. Unde conſtat in circulo omnes ad peripheriam normales in centro concurrere, adeoque tangentem radio normaliter inſiſtere.

COROLL. IV.

119. In Verſoria quoniam eſt $xyy + aax = ayy$, erit $2xydy + yydx + aadx = 2aydy$, quare $\frac{ydy}{dx}$
$$= \frac{yy + aa}{2a - 2x} = \frac{aax}{(2a - 2x)(a - x)} + \frac{aa}{2a - 2x} = \frac{a^3}{2aa - 4ax + 2xx}.$$
Idem eruitur ex altera Analogia, quam ſupra indicavimus.

Quando punctum G cadit in vertice A, fit $a - x = a$, & ſubnormalis eſt $= \frac{1}{2}a$. *Fig. 7.*

COROLL. V.

120. Denique in Conchoide, cujus æquatio $y = \frac{(b - x)\sqrt{(2ax - xx)}}{a - x}$, erit normalis $\frac{ydy}{dx} =$
$$\frac{(aab - 3aax + 3axx - xxx)(b - x)}{(a - x)^3}.$$

SCHOL.

121. Vir. Cl. SCHOOTENIUS in ſecunda editione Comment. Geometriæ Carteſianæ lib. 2. expeditam docet normalis conſtructionem pro Conchoide Nicomedea A E; nempe ex polo P ducit perpendicularem PN ad ramum PT ſecantem regulam BF in H, & ex H ipſi regulæ normalem HN occurrentem alteri *Fig. 9.*

Digitized by Google

alteri ad ramum normali PN, & junctam NT ait eſſe Conchoidi perpendicularem. Verum id longe univerſalius proponi poteſt hoc modo. Eſto quælibet regula recta, vel curva BF, juxta quam polo P deſcribatur Conchois AE, ita nempe, ut intervalla BA, HT ſint ſemper eidem quantitati æqualia, atque HN normalis regulæ, vel ejus tangenti in H, ſi regula fuerit curva, conveniat cum PN normali ad ramum in puncto N, juncta NT erit normalis Conchoidi AT, ejuſve in T tangenti.

PROBL. XXVII.

Fig. 29. 120. Si quævis figura AEFG circa ſuum axem AD rotata producat ſolidum GHMA quod ſecetur plano EHK ad baſim normali faciente ſectionem ELHK cujus baſis HK normaliter ſecetur a diametro baſis GDM in K invenire rationem ſubnormalium in curvis ELH, EFG, quæ reſpondeant ordinatis IL, FC.

Sit $BC = EI = x$, $IL = y$, $FC = v$, $EB = b$, quoniam eſt $CF^2 = CL^2 = IL^2 + IC^2 = IL^2 + EB^2$, erit $vv = yy + bb$, quare $2vdv = 2ydy$, & $vdv = ydy$; atqui ſubnormalis curvæ EFG eſt $\frac{vdv}{dx}$, & ſubnormalis curvæ ELH eſt $= \frac{ydy}{dx}$, ergo quoniam $\frac{vdv}{dx} = \frac{ydy}{dx}$, ſubnormales hujuſmodi curvarum erunt æquales.

PROBL. XXVIII.

Fig. 30. 121. Sit quælibet figuræ ſpecies DFH, & ubique fiat $EG^2 = EF^2 + BC^2$ [ſumta ſcilicet ad arbitrium conſtanti BC] invenire rationem ſubnormalium in figuris CG, DFH reſpondentium ordinatis EG, EF.

Sit

Sit $EF = y$, $EG = v$, $BC = a$, erit $yy + aa = vv$, & $2ydy = 2vdv$, quare & $\frac{ydy}{dx} = \frac{vdv}{dx}$, ſcilicet ſubnormales æquantur.

PROBL. XXIX.

122. Invenire formulam generalem ſubnormalis in ang. coordinatarum obtuſo, vel acuto.

Sint omnia ut in probl. 25., & ponatur NG normalis ad curvam AG, & GF normalis diametro TN. Et quoniam triang. OPH, GFS ſunt ſimilia, erit $HO : OP = SG : GF$, $m : n = y : \frac{ny}{m} = GF$,

& $OH : HP = GS : SF$, $m : p = y : \frac{py}{m} = SF$.

Cum autem ſimilia ſint triang. OGP, GFN, erit $GP : PO = GF : FN$, $\frac{mdx \pm pdy}{m} : \frac{ndy}{m} = \frac{ny}{m} :$

$\frac{n^2 ydy}{m^2 dx \pm mpdy} = FN$, atque hinc $SN =$

$$\frac{n^2 ydy}{m^2 dx \pm mpdy} \pm py = \frac{mn^2 ydy + m^2 pydx \pm mp^2 dy}{m^3 dx \pm m^2 pdy}$$

$$= \frac{n^2 ydy + mpydx \pm p^2 dy}{m^2 dx \pm mpdy} = \frac{mydy \pm pydx}{mdx \pm pdy};$$

nam eſt $m^2 = n^2 + p^2$ [§. 114.]

PROBL. XXX.

125. Invenire formulam generalem normalis in ang. coordinatarum recto.

Sint omnia ut probl. 5. Et quoniam triang. MRr, PNM ſunt ſimilia, erit $MR : Mr = PM :$

MN, $dx : V[dx^2 + dy^2] = y : \frac{y}{dx} V[dx^2 + dy^2]$ quæ eſt formula normalis MN.

COROLL. I.

126. Quoniam igitur in parabola eſt $2ydy = adx$, erit $dy^2 = \frac{aadx^2}{4yy}$, hinc ſubſtituendo erit $\frac{y}{dx}$ $V(dx^2 + dy^2) = V(ax + \frac{1}{4}aa)$.

COROLL. II.

127. In circulo quoniam eſt $2ydy = adx - 2xdx$, erit $dy^2 = \frac{aadx^2 - 4axdx^2 + 4xxdx^2}{4yy}$; quare ſubſtituendo erit $\frac{y}{dx} V[dx^2 + dy^2) =$

$$V\frac{(4ax - 4xx + aa - 4ax + 4xx)}{4} = \frac{1}{2}a.$$

DEF. V.

128. Aſymptotus curvæ eſt linea ſive recta, ſive curva, quæ quo magis producitur, eo magis continue accedit ad curvæ crus, cum quo tandem coincidit.

SCHOL.

129. Sint duæ ſimiles, & æquales curvæ aſymptoticæ DGL, CKN, aut AGI, BKN; quoniam ducta ubilibet EGK eſt FG = HK, dempta, vel addita HG, fiet GK = FH; quare ſpatium infinite longum curvis illis interceptum erit = parallelogrammo ADCB. Jam vero ducta diametro parallelogrammi DB, ſi ipſi FG fiat = FI = HK, curva DIM, vel BIM haberi poterit pro diametro parallelogrammi curvilinei

Fig. 31.

curvilinei ab hiſce curvis intercepti, earumque ſpatium bifariam dividet; Enimvero eſt F I = E G; hinc ablata, vel appoſita communi F G, G I = E F, & reliqua I K = F H, quare ſpatium L G D I M in prima fig. = triang. A D B, & reliquum M I C K N = triang. B D C; itemque in ſecund. fig. L G A B I M = triang. D A B, & reliquum ſpatium M I B K N = triang. B D C; unde curva M I bifariam dividit ſpatium curvis L G, N K interceptum. Quoniam autem interceptæ in prima fig. M I D C K N = ordinatis triang. D B C, videlicet I K = F H, dictæ curvæ I M, K N concurrent ad infinitam diſtantiam cum aſymptoto, quippe earundem intervallum fit = puncto B, quod eſt apex ejuſdem triang. Pariter in ſecund. fig. curvæ A G L, B I M concurrent in aliquod punctum in aſymptoto infinite producta; ibi enim in aſymptoto abit in punctum D ordinata E F triang. D A B, cui æqualis eſt intercepta G I. Curvæ igitur aſymptoticæ cum aſymptoto ipſa ad infinitam diſtantiam reipſa conveniunt.

PROBL. XXXI.

130. Curvarum algebraicarum aſymptotos determinare. Sit ex: gr. hyperbola vulgaris *Apollonii* A M r, cujus æquatio, poſita ſemidiametro C A = a, A P = x, P M = y, dimidio latere recto = b, erit $ayy = 2abx + bxx$. Patet ex §. 35. ejuſdem ſubtangentem P T eſſe $= \frac{DPA}{CP} = \frac{2ax + xx}{a + x}$, unde ſi auferatur A P $= x$, fiet reliqua A T $= \frac{CAP}{CP} = \frac{ax}{a + x}$. Sed ubi A P, vel C P abit in infinitum, ipſa C A fit infinite parva, vel nulla relate ad A P; quare rectang. $\frac{CAP}{CP}$ idem eſt cum rectang. $\frac{CAP}{AP} = \frac{ax}{x} = a =$ C A; proindeque in caſu aſymptoti recta A T fit = C A, ſcilicet aſym- Fig. 32.

 ptotus

ptotus oritur ex centro hyperbolæ. Nunc ducta AF parallela ordinatis erit $TA : AF = dx : dy$. At vero ſi x ponatur infinita, erit in æquatione curvæ terminus $2abx$ infinite minor termino bxx, ac proinde nullus, quare tunc fiet æquatio $ayy = bxx$, vel $y\sqrt{a} = x\sqrt{b}$, cujus differentia eſt $dy\sqrt{a} = dx\sqrt{b}$: Eſt igitur $dx : dy = \sqrt{a} : \sqrt{b}$; ſed $dx : dy = TA : AF$, ergo $TA : AF = \sqrt{a} : \sqrt{b}$, nempe $TA : AF$ in ſubduplicata ratione $a : b$. Sed in caſu noſtro $TA = CA = a$, ergo ipſa $AF = \sqrt{ab}$ nempe eſt media proportionalis inter ſemilatus rectum, & dimidium tranſverſi, ut *Apollonius* demonſtravit; Quare ſi jungantur puncta C, F, habebitur poſitio rectæ, quæ eſt hyperbolæ aſymptotus.

Eademque ratione determinantur aſymptoti in qualibet alia curva.

SCHOL.

139. Huc uſque methodum docuimus, qua determinari poſſint aſymptoti in æquationibus, in quibus non adeſt rectang. xy. Verum enim vero cum, manente rectang. xy in æquatione, modus ſit aliquanto immutandus, non abs re erit, qua ratione id fieri poſſit, demonſtrare. Sit igitur æquatio $y^3 - x^3 = axy$, erit $3yydx - 3xxdx = axdy + aydx$; hinc $\frac{ydx}{dy} = \frac{3y^3 - axy}{3ax - ay}$, & $AT = \frac{ydx}{dy} - x = \frac{3y^3 - axy - 3x^3 - axy}{3xx + ay} = \frac{+axy}{3x^2 + ay}$, ſi loco $3y^3 - 3x^3$ ejus valor $3axy$ ſufficiatur. Ex quo patet tolli hinc non poſſe terminos, qui infinite parvi ſunt, adeoque negligendi præ majoribus, ut ſupra factum fuit. Ponatur itaque AP infinite protenſa, fiet PM infinita, tangens vero TM tunc erit aſymptotus, & recta AT fiet AC, atque adeo ipſæ AC, AF aſymptoti

ptoti poſitionem determinabunt. Jam vero ſi dicatur $AT = t$, erit $t = \frac{axy}{3x^2 + ay}$, hinc $3txx + aty = axy$, & $y = \frac{3txx}{ax - at} = \frac{3tx}{a}$, ſiquidem in infinitum abeunte x, ipſa at evaneſcit præ ax, adeoque negligi tuto poteſt. Hoc vero caſu $y^3 - x^3 = \frac{27t^3x^3}{a^3} - x^3 = \frac{27t^3x^3 - a^3x^3}{a^3}$, & $axy = \frac{3atxx}{a} = 3txx$, ergo $\frac{27t^3x^3 - a^3x^3}{a^3} = 3txx$, & $27t^3x^3 - a^3x^3 = 3a^3txx$, ſive $27t^3x^3 = 3a^3tx^2 + a^3x^3$; ſed productum $3a^3tx^2$ evaneſcit præ $27t^3x^3$, & a^3x^3, quare $27t^3 = a^3$, & $t = AT = \frac{1}{3}a$.

Ut autem inveniatur AF, quo punctum alterum determinetur, per quod duci debeat aſymptotus, concipiatur triang. MRr infinite diſtans, & mente intelligatur producta RM in G, erit $AF = y - \frac{xdy}{dx}$ (§. 48.). At vero ex differentia datæ æquationis eſt $dy = \frac{3xxdx + aydx}{3yy - ax}$, ergo $y - \frac{xdy}{dx} = \frac{3y^3 - 2axy - 3x^3}{3yy - ax} = \frac{axy}{3yy - ax}$, quæ dicatur s;

Quare

Quare $3syy - asx = axy$, & $x = \frac{3syy}{ay + as} = \frac{3sy}{a}$; quippe abeunte in infinitum tum x, tum y quantitas as evaneſcit præ ay, & $ax = 3sy$, unde $s = \frac{ax}{3y}$.

Sed $y = \frac{3tx}{a}$, & $3y = \frac{9tx}{a}$, quare $\frac{ax}{3y} = \frac{a^2x}{9tx} = \frac{a^2}{9t}$. Sed $t = \frac{1}{3}a$, ergo $\frac{a^2}{9t} = \frac{1}{3}a = s = AF$.

Hinc ſi ſumantur AF, & $AC = \frac{1}{3}a$, juncta recta CF, & indefinite producta erit curvæ AM aſymptotus.

COROLL.

132. Unde conſtat methodum, qua ducuntur aſymptoti eandem eſſe cum methodo, qua tangentes determinantur, hoc tamen diſcrimine, quod ad ducendas tangentes ſatis eſt invenire punctum T, unde duci poſſit recta TM, in ducendis vero aſymptotis punctum aliud F determinari debet ob infinitam diſtantiam punctorum T, M, qua fit, ut eadem puncta nulla recta linea conjungi poſſint.

PROBL. XXXII.

133. Data relatione inter abſciſſam AB, & ſubnormalem NB invenire æquationem, quæ curvæ AC naturam definiat. *Fig. 33.*

Quoniam eſt $NBT = BC^2$ multiplicentur omnes termini NB denotantes per terminum AB prius in diverſos ignotos numeros multiplicatum, & productum ponatur $= BC^2$.

Ratio regulæ colligitur ex methodo, qua Cl. S. Lucius tangentes ducit * Sit igitur $AB = x$, $BC = y$,

numeri

* *Vid. Memoirs of the Royal Society.*

numeri ignoti p, q, r, $ſ$, t, quantitates vero notæ, & determinatæ a, b, c, e.

Sit I. $BN = b$, fiat $bpx = yy$, erit æquatio curvæ quæſitæ ad parabolam.

Sit II. $BN = b + x$, erit $bpx + qxx = yy$.

Sit III. $BN = \frac{xx}{a} + c$, erit $\frac{rx^3}{a} + c\mathit{ſ}x = yy$.

Sit IV. $BN = \frac{x^4}{e^3} + \frac{x^3}{ee} + \frac{xx}{e} + x$, erit $\frac{tx^5}{e^3} + \frac{px^4}{ee} + \frac{qx^3}{e} + rxx = yy$.

Sit V. $BN = \frac{b^3}{xx}$, erit $\frac{bp^3}{x} = y$, ſi ducatur $\frac{b^3}{xx}$ in px.

Atque hæ ſunt æquationes reſpective determinantes harum curvarum naturam, modo determinentur incognitæ quantitates p, q, r, $ſ$, t &c. quod qua ratione fiat ſequenti probl. demonſtrabitur.

PROBL. XXXIII.

134. Quantitates incognitas ſuperioris methodi determinare.

Inveniatur curvæ tangens per æquationem ſupra adinventam, ejuſque valor cum valore dato conferatur, ſingulos terminos cum ſingulis comparando, & ſic quantitates ignotæ p, q, r, $ſ$, t &c. determinabuntur.

Ex. gr. cum ſit $be : eC = CB : BN$, $dx : dy = y : \frac{ydy}{dx} = BN$, ſumatur differentia æquationis inventæ, quo valor ipſius BN, a differentiis liber inveniatur, & cum valore ejuſdem antea dato comparetur.

Sit

Digitized by Google

Sit igitur I. $bpx = yy$, erit $bp\,dx = 2y\,dy$, & $\frac{1}{2}bp = \frac{y\,dy}{dx}$; ſed prius erat eadem $BN = b$, ergo $b = \frac{1}{2}bp$, & $p = 2$, proindeque æquatio illa $bpx = yy$ vertetur in $2bx = yy$.

II. Quoniam eſt $bpx + qxx = yy$, erit $bp\,dx + 2qx\,dx = 2y\,dy$, & $\frac{bp + 2qx}{2} = \frac{y\,dy}{dx}$. Sed prius erat $BN = b + x$, quare $b = \frac{bp}{2}$, & $p = 2$; item $x = qx$, hinc $q = 1$. Suffectis itaque his valoribus fiet æquatio $2bx + x^2 = y^2$.

III. Sit $\frac{rx^3}{a} + csx = yy$, erit $\frac{3rxx\,dx}{a} + cs\,dx = 2y\,dy$, & $\frac{y\,dy}{dx} = \frac{3rxx}{2a} + \frac{1}{2}cs$, hinc $\frac{3rxx}{2a} = \frac{xx}{a}$, & $r = \frac{2}{3}$; item $\frac{1}{2}cs = c$, & $s = 2$, quare habebitur æquatio $\frac{2x^3}{3a} + 2cx = yy$; Et ſic de ceteris.

SE-

SECTIO II.

De Calculo Differentiarum ſecundarum.

CAP. I.

De Natura Calculi Differentiarum ſecundarum.

DEFINITIO I.

135. Calculus *differentiarum ſecundarum* eſt methodus, qua differentiarum primarum differentiæ determinantur, ſeu quantitatum variabilium differentiæ ſecundæ ex primis inveniuntur.

DEF. II.

136. Differentia, ſive quantitas infinite parva, qua vel augetur, vel minuitur quolibet inſtanti quantitas variabilis x, vel y, vel v &c. tunc dicitur *differentia conſtans*, cum eſt ſibi ſemper æqualis, ſcilicet neque incrementum, neque decrementum ſuſcipit: Ita ex. gr. ſi hæc quantitas ſit dx, vel dy, vel dv &c. ea nempe, quæ neque incrementum, neque decrementum ſuſcipiat, eam dicimus conſtantem. Contra vero eandem vocamus *variabilem*, ſi quolibet inſtanti incrementum aliquod, vel decrementum ſuſcipiat infinite parvum relate ad ſe ipſam.

DEF. III.

137. Incrementum, vel decrementum infinite parvum, quod quantitas infiniteſima, ſive differentia variablis dx, vel dy, vel dv &c. ſuſcipit quolibet inſtanti, dicitur *differentia differentiæ*, ſive *differentia ſecunda*, Quod ſi hæc differentia ſecunda ſit pariter variabilis, quantitas infinite parva relate ad ſe ipſam, qua augetur, vel minuitur quolibet inſtanti, dicitur *differentia tertia*, & ſic deinceps.

Digitized by Google

COROLL.

138. Quoniam vero differentiæ infinite parvæ quantitatum variabilium exprimuntur per notam, ſeu litteram d [§. 4.] differentiæ differentiarum, ſive differentiæ ſecundæ earundem quantitatum exprimentur per duplicem d, nempe dd. Ita differentia ipſius dy, ſive differentia ſecunda ipſius y erit ddy, vel d^2y, eademque ratione differentia ipſius ddy erit $dddy$, vel d^3y, quæ dici poterit differentia tertia ipſius y, & ſic deinceps.

PROBL. XXXIV.

139. Quantitatum ex differentiis compoſitarum differentias, ſcilicet differentiarum primarum differentias invenire.

Eruatur I. ex circumſtantiis caſuum ſpecialium quænam ſint quantitates conſtantes, ut earum differentiæ ſecundæ ſint $=0$, quænam vero variabiles. II. Loco quantitatis, quæ ſumenda eſt ut conſtans ponatur quantitas quæpiam conſtans, nempe p, vel q, vel r &c. III. Loco quantitatis variabilis ponatur nova ignota; IV. tandem, & ultimo ſumantur differentiæ juxta regulas, quas Cap. I. Sect. I. tradidimus, ubi de ſumendis differentiis quantitatum variabilium agebamus. Rem illuſtremus exemplis.

Sit igitur quærenda differentia quantitatis $\frac{cxdx}{ydy}$.

Si ſumenda ſit dx conſtans, ejuſdem loco ponatur p, & loco dy ponatur z, migrabit itaque propoſita quantitas in hanc $\frac{cxp}{yz}$. Fiat $\frac{cxp}{yz}=v$, erit $cxp=yzv$, & ſumtis differentiis $cpdx=yzdv+yvdz+zvdy$, ſive $cpdx-zpdy-yvdz=yzdv$. Reſtituatur nunc priſtinus valor quantitatum, quas in alias commutavimus, nempe dx loco p, dy loco z, & ddy loco dz, & erit $\frac{cdx^2}{1}-\frac{cxdxdy}{y}-\frac{cyxdxddy}{ydy}=$

$vdydv$, ſive $\frac{cydydx^2 - cxdxdy^2 - cyxdxddy}{yydy^2} = dv$.

Sit determinanda differentia quantitatis dy^2. Quoniam dy^2 eſt quadratum ipſius dy, fiat $dy = v$, erit $dy^2 = vv$, hinc differentia ipſius $dy^2 =$ differentiæ ipſius $vv = 2vdv$. Loco v, & dv ſufficiantur eorundem valores dy, & ddy, & erit differentia quantitatis $dy^2 = 2dyddy$.

Sit invenienda differentia quantitatis $\frac{V(dx^2 + dy^2)}{dx}$ & dx ponatur conſtans $= p$: Ergo data quantitas vertetur in $\frac{V(p^2 + dy^2)}{p}$, quæ fiat $= z$. Erit igitur $V(p^2 + dy^2) = pz$, & $pp + dy^2 = ppzz$, ſumtiſque differentiis $2dyddy = 2ppzdz$, ſive $\frac{dyddy}{ppz} = dz$. Reſtituantur valores pp, & z, eritque $\frac{dyddy}{dxV(dx^2 + dy^2)} = dz$.

Sit pariter veſtiganda differentia quantitatis $\frac{vV(dx^2 + dy^2)}{dy}$. Fiat dy, quæ hic adſumitur ut conſtans, $= p$, & propoſita quantitas vertetur in $\frac{vV[dx^2 + dv^2]}{p}$, quæ ponatur $= z$. Erit ergo $vV[dx^2 + dv^2] = pz$, & $vvdx^2 + vvdv^2 = ppzz$, quare $2vdvdx^2 + 2vvdxddx + 2vdv^3$

$+ 2vvdvddv = 2ppzdz$, &

$$\frac{vdvdx^2 + vvdxddx + vdv^3 + vvdvddv}{ppz} =$$

dz. Loco pp, & z ſufficiantur eorum valores fiet

$$\frac{vdvdx^2 + vvdxddx + vdv^3 + vvdvddv}{vdy\sqrt{[dx^2 + dv^2]}} =$$

$$\frac{dvdx^2 + vdxddx + dv^3 + vdvddv}{dy\sqrt{(dx^2 + dv^2)}} = dz.$$

Eodem plane modo ſumuntur differentiæ tertiæ, quartæ, quintæ &c. quantitatum variabilium, nimirum ſi iiſdem legibus, ac primæ tractentur.

NOTA

140. Adnotandum hic eſt in quovis æquationis membro differentialia ſimul ſumta eundem dimenſionum numerum efficere; nempe ſi in uno membro adfuerit dy^3, aderit in aliis aut dx^3, aut $dxdy^2$, aut $dydx^2$, quod patet.

SCHOL.

141. Lubet autem hic inquirere in hujuſmodi differentias ſecundas, tertias, quartas &c. ordinatarum ad curvam, in qua ipſæ dx ponuntur conſtantes, & methodum geometricam adferre ex Cl. GUIDONE GRANDIO depromtam * qua differentiæ ſecundæ ex primis, tertiæ ex ſecundis, ex tertiis quartæ, idque ſine ullo limite, determinentur. Si ad axem AG conſtituta ſit figura ABC cujus ducatur tangens CT, & fiat ſemper ut ſubtangens GT ad GC, ita recta aliqua conſtans AK ad GN, orietur nova curva QLMN, in

* *Append*: 2. *De transform*: *curv*. *Coroll*. 8. *Theor*. 1.

Fig. 34.

in qua primæ differentiæ mM ordinatarum NG proportione reſpondebunt ſecundis differentiis ordinatarum CG in curva priore. Si enimvero ducatur agM infinite proxima ipſi CGN, & agatur Nm, parallela axi AG, cum ſit $Cc : ca = CG : GT = GN : AK$, erit rectang. $Cc . AK =$ rectang. $ca . GN$; Quare ſi AG ponatur $= x$, $GC = y$, $Cc = dy$, $Gg = dx$, $GN = v = mM\, dv$, erit $ady = vdx$; quapropter ſumtis differentiis, & dx poſita conſtante, erit $addy = dvdx$, ſive $\frac{addy}{dx} = dv$: Hinc cum a, & dx ſint conſtantes, dv proportionalis erit ipſi ddy, nempe differentia prima ordinatæ GN proportionalis erit differentiæ ſecundæ ordinatæ GC; Eſt enim $a : dx = dv : ddy$; Sed $a : dx = v : dy$, nam $ady = vdx$, ergo $v : dy = dv : ddy$, GN ad Cc primam differentiam ipſius CG, ut prima differentia mM ipſius GN ad differentiam ſecundam ipſius CG.

Eadem ratione eſſet ſecunda differentia ipſius GN proportionalis tertiæ differentiæ ipſius CG, & ſic deinceps.

CAP.

CAP. II.

De Uſu Calculi Differentiarum diverſi ordinis in quæſtionibus de Maximis, & Minimis.

DEFINITIO.

142. Quando natura variabilis quantitatis ea eſt, ut continuo creſcat ſine limite, aut decreſcat, donec penitus evaneſcens fiat = o, tunc nulla ejuſdem magnitudo poteſt adſignari, quæ dici valeat maxima, vel minima; Quando vero limes aliquis eſt conſtitutus, quem variabilis quantitas continuo creſcens, vel decreſcens prætergredi non poteſt, & terminus eſt adſignabilis, ubi eadem quantitas hujuſmodi limitem attingit, quantitas illa ſub hunc limitem dicitur *maxima*, vel *minima*. Sit curva ABD, cujus ordinatæ FG, *fg* ſint ſibi mutuo parallelæ, in eaque creſcentibus abſciſſis ordinatæ ad certum uſque, ſtatutumque limitem, nempe ad ordinatam BC creſcant, aut decreſcant, mox ultra hanc ordinatam, quæ creſcebant decreſcant identidem, vel nullæ ſint, & contra creſcentibus abſciſsis ordinatæ decreſcant uſque ad aliquod punctum, ſive ad ordinatam BC, poſt quam aut creſcant, aut nullæ ſint amplius ordinatæ, locus in quo ordinatim poſitæ permutantur, ſcilicet ordinata BC dicitur *maxima*, vel *minima* quantitas; quæſtio vero de hujuſmodi quantitatibus inveniendis quæſtio de *maximis*, vel *minimis* appellatur. *Fig. 25. & 35.*

COROLL.

143. Hinc patet hujuſmodi quæſtiones eo redire, ut detegatur cujuſnam longitudinis eſſe debeat abſciſſa, quo eidem reſpondens ordinata ſit omnium maxima, vel minima.

THEOREMA II.

144. Si curva AGB concavitatem axi obvertat, ejuſque ordinatæ ſemper creſcant uſque ad limitem CB,

CB, differentia ordinatarum ſub hunc limitem fiet nulla.

Cum enim in puncto B tangens BH fiat baſi ACD parallela (§. 76.) adeoque concipi poſſit velut ad infinitam diſtantiam cum eadem concurrens, ſubtangens $\frac{ydx}{dy}$ abibit in infinitum; quoniam vero eſt $dy : dx = y : \frac{ydx}{dy}$, erit $dy = 0$; ratio ſiquidem ipſius y ad ſubtangentem eſt ratio finiti ad infinitum, ſeu eadem cum ratione $0 : 1$, dx vero conſtans, ac proinde dy nulla relate ad dx.

COROLL.

145. Hinc quoties differentia ordinatæ, nempe dy fiet $= 0$, emerget valor ordinatæ omnium maximæ.

THEOREMA III.

146. Si curva convexitatem axi obvertat, & ordinatæ decreſcant uſque ad limitem BC, in eo differentia ordinatarum fiet nulla.

Eodem plane modo demonſtratur, quo præcedens Theorema.

COROLL.

147. Conſtat hinc, quod ſi dy ponatur $= 0$, determinabitur valor indeterminatæ y omnium minimæ.

THEOREMA IV.

148. Si evaneſcente differentia prima ordinatæ, ejuſdem differentia ſecunda ſit poſitiva, tunc ordinata eſt minima, maxima vero, ſi ejuſdem differentia ſecunda ſit negativa. *Fig. 35.*

Patet hoc ex Mac-Laurino *. Sit enimvero ordinata $CB = E$, $CK = x$, erit ordinata $KG = E + \frac{dEx}{dx} + \frac{ddEx^2}{2dx^2} + \frac{dddEx^3}{6dx^3} + \&c.$ Ponatur enim y exprimi per ſeriem hujuſce formæ $A + Bx + Cx^2 + Dx^3 + \&c.$ in qua quantitates A, B, C, D &c. re-

* *Cap. 5. lib. 2. art. 858. Treatiſe of Fluxions. Eundem vide Cap. 9. lib. 1.*

repræſentant coefficientes invariabiles, quorum quilibet ſupponi poteſt evaneſcere; Ubi x evaneſcit, fit E valor ipſius y, & dE, ddE, $dddE$ ſint tunc reſpectivi valores quantitatum dy, ddy $dddy$ &c. Tunc

$$y = E + \frac{dEx}{dx} + \frac{ddEx^2}{1.2dx} + \frac{dddEx^3}{1.2.3dx^3} + \&c.$$

Patet lex continuationis ſeriei; Nam quoniam $y = A + Bx + Cx^2 + Dx^3 + \&c.$ ſequitur, quod ubi $x = o$, A fit $= y$. Sed [hypoth.] tunc $E = y$, ergo $A = E$. Sumantur differentiæ, erit $dy = Bdx + 2Cxdx + 3Dx^2dx + \&c.$ Quare $\frac{dy}{dx} = B + 2Cx + 3Dx^2 + \&c.$ Sed ubi $x = o$, fit $B = \frac{dy}{dx} = \frac{dE}{dx}$; Rurſus ſumantur differentiæ erit $ddy = 2Cdx^2 + 6Dxdx^2 + \&c.$ & $\frac{ddy}{dx^2} = 2C + 6Dx^2 + \&c.$ Sit $x = o$, erit $2C = \frac{ddy}{dx^2} = \frac{ddE}{dx^2}$, & $C = \frac{ddE}{2dx^2}$. Iterum ſumantur differentiæ, erit $dddy = 6Ddx^3$, & $\frac{dddy}{dx^3} = 6D + \&c.$ poſitoque $x = o$, erit $D = \frac{dddy}{6dx^3}$, adeoque $y = A + Bx + Cx^2 + Dx^3 + \&c.$ erit $= E + \frac{dEx}{dx} + \frac{ddEx^2}{1.2dx^2} + \frac{dddEx^3}{1.2.3dx^3} + \&c.$ Hoc demonſtrato ſumatur CF ad alteram plagam ipſius C $=$ CK, tunc ordinata FA erit $= A - Bx + Cx^2 - Dx^3$

$$Dx^3 + \&c. = E - \frac{dE\,x}{dx} + \frac{ddE\,x^2}{2dx^2} - \frac{dddE\,x^3}{6dx^3} + \&c.$$

Ponamus nunc $dE = o$, tunc $KG = E * + \frac{ddE\,x^2}{2dx^2}$ &c.

& $FA = E * + \frac{ddE\,x^2}{2dx^2} - \&c.$: Quare ſi diſtantiæ CK, CF ſint minimæ, ordinatæ KG, FA excedent ambæ ordinatam CB, ſi ddF ſit poſitiva, ſi vero fuerit negativa, erunt ambæ minores ipſa CB. Quod ſi præter dE evaneſcat etiam ipſa ddF, & $dddE$ non evaneſcat ſimul altera ex ordinatis KG, FA erit > CB, altera vero minor, ita ut in hoc caſu ordinata ſit nec maxima, nec minima.

Generatim ſi differentia prima ordinatæ cum differentiis diverſorum ordinum ſubſequentium evaneſcat, ordinata eſt minima, vel maxima, quando numerus illarum omnium differentiarum, quæ evaneſcunt eſt 1, 3 5, vel quilibet numerus impar; ordinata nempe eſt minima, quando differentia proxime ſequens illam, quæ evaneſcit eſt poſitiva; maxima vero, quando hæc differentia eſt negativa. Hoc patet ex comparatione valorum ipſarum KG, FA. Sed ſi numerus differentiarum omnium ordinatæ primi, & ſubſequentium ſucceſſivorum ordinum, quæ evaneſcunt, ſit numerus par, ordinata eſt nec maxima, nec minima.

Hæc valent etiam pro radiis, qui a dato puncto ducuntur ad curvam. *

* *Eodem Cap. 9. art. 277.*

PROBL. XXXV.

149. Determinare valores ignotæ x, ubi valor quantitatis $3x^4 - 28ax^3 + 84aaxx - 96a^3x + 48c^4$ ſit maximus, vel minimus.

Sumatur differentia propoſitæ quantitatis, & æquetur nihilo, erit $12x^3dx - 84ax^2dx + 168aaxdx - 96a^3dx = 0$, vel $x^3 - 7axx + 14aax - 8a^3 = 0$; cujus æquationis radices ſunt a, $2a$, $4a$.

SCHOL. I.

150. Ut autem cognoſcatur, quænam ex hiſce radicibus præbeat maximum, quæ vero minimum, ſumantur differentiæ ſecundæ ejuſdem quantitatis, erit $3xxdx^2 - 14axdx^2 + 14aadx^2 = 0$; Nunc loco x ſufficiatur a, habebitur $3aadx^2 - 14aadx^2 + 14aadx^2 = 0$; Quare cum differentiæ ſecundæ in hoc caſu ſint poſitivæ, radix a præbebit minimum. Subſtituatur nunc loco x radix $2a$, habebitur $12aadx^2 - 28aadx^2 + 14aadx^2 = 0$; cum itaque differentiæ ſecundæ in hoc caſu ſint negativæ, radix $2a$ præbebit maximum. Denique loco x ſufficiatur $4a$, emerget $48aadx^2 - 56aadx^2 + 14aadx^2 = 0$, proindeque cum differentiæ ſecundæ in hoc caſu ſint poſitivæ, radix $4a$ præbebit minimum. Ulterius ut detegatur, utra ex duabus minimis radicibus præbeat minimum valorem, ſufficiatur ea in propoſita quantitate, & quæ minimum præbebit valorem, erit minima. Quoniam itaque ſuffecta a habetur $3a^4 - 28a^4 + 84a^4 - 96a^4 + 48c^4 = 48c^4 - 37a^4$, ſuffecta vero $4a$ habetur $48c^4 - 64a^4$, patet radicem $4a$ eſſe minimam, & minimum valorem propoſitæ quantitatis.

SCHOL. II.

151. In omni quæſtione de maximis, & minimis quælibet quantitas indeterminata, & ex aliis, ut libuerit, compoſita concipienda eſt ad inſtar ordinatæ ad aliquam

curvam,

curvam, eaque ponenda eſt æqualis alteri indeterminatæ, ea quidem lege ut ſi quæſtio proponatur de minima quantitate invenienda figura ad conſtructionem excogitata cum minima ordinatarum ſit deſcribenda, ſecus vero, ſi quæſtio inſtituatur de maxima; ex quo fiet ut ubi cadit maxima, vel minima ordinata hujuſmodi curvæ, ibi maxima, vel minima quæſita quantitas habeatur.

PROBL. XXXVI.

152. Invenire maximam, vel minimam ordinatam in curva æquationis $cy = aa + 2bx - xx$.

Erit ergo $cdy = 2bdx - 2xdx$, & $cddy = -2dx^2$; ergo ordinata eſſe debet maxima (§.148.). Quoniam vero $cdy = 2bdx - 2xdx$, erit $\frac{dy}{dx} = \frac{2b - 2x}{c}$. Ponatur $dy = 0$, erit $\frac{2b - 2x}{c} = 0$, & $x = b$, & $y = \frac{aa + bb}{c}$.

PROBL. XXXVII.

153. Determinare maximam, vel minimam ordinatam in curva, ad quam $ay^3 = abxx - bx^3$.

Erit igitur $3ayydy = 2abxdx - 3bxxdx$, & $dy = \frac{2abxdx - 3bxxdx}{3ayy}$; hinc $ddy = -\frac{2bbxxdx^2}{9y^5}$; ordinata igitur erit maxima; cum autem ſit $dy = \frac{2abxdx - 3bx^2dx}{3ayy}$ erit $\frac{dy}{dx} = \frac{2abx - 3bxx}{3ayy}$. Ponatur $dy = 0$, erit $2abx - 3bx^2 = 0$, & $x = \frac{2}{3}a$, proindeque $y = \sqrt[3]{\frac{4aab}{27}}$.

PROBL. XXXVIII.

154. Determinare maximam, vel minimam ordinatam in curva, ad quam $cy = 2xx - 2bx + 2aa + bb$.

Eſt $cdy = 4xdx - 2bdx$, & $ddy = \frac{4xdx}{c}$;

ergo ordinata eſt minima. Quod ſi ponatur $dy = 0$, erit $4x = 2b$, & $\frac{1}{2}b = x$, & $y = \frac{\frac{1}{2}bb + 2aa}{c}$.

PROBL. XXXIX.

155. Invenire maximam, vel minimam ordinatam in curva æquationis $ay = b^2 - 2cx + xx$.

Erit $ady = 2cdx + 2xdx$, & $ddy = \frac{2dx^2}{a}$;

ergo, quoniam poſito $dy = 0$, fit $x = cy$ erit minima ordinata $= \frac{bb - cc}{a}$, ſi b ſit $> c$.

PROBL. XL.

156. Invenire maximam, vel minimam ordinatam in curva, ad quam $y^4 = ax^3 - x^4$.

Erit ergo $4y^3dy = 3axxdx - 4x^3dx$, &

$dy = \frac{3axxdx - 4x^3dx}{4y^3}$; hinc poſito $dy = 0$, erit

$3axx = 4x^3$, & $x = \frac{3}{4}a$, ordinata vero huic abſciſſæ reſpondens eſt omnium maxima, cum ejus differentia ſecunda ſit negativa.

PROBL. XLI.

157. Invenire maximam ordinatam in curva, ad quam $x^4 + 2x^2y^2 + y^4 - a^2x^2 + a^2y^2 = 0$.

Erit

Erit igitur $4x^3dx + 4xxydy + 4yyxdx + 4y^3dy - 2aaxdx + 2aaydy = 0$, & $dy = \frac{2aaxdx - 4x^3dx - 4yyxdx}{4xxy + 4y^3 + 2aay} = 0$, hinc $2aax - 4x^3 = 4yyx$, hoc eſt $\frac{1}{2}aa - xx = yy$. Loco yy ſufficiatur in æquatione propoſita ipſius valor, erit $- aaxx + \frac{1}{2}a^4 - aaxx = 0$, ſive $\frac{3}{4}a^4 - 2aaxx = 0$, & $\frac{3}{8}aa = xx$, ſive $x = \sqrt{\frac{3}{8}aa}$, quæ eſt abſciſſa maximam determinans ordinatam in data curva; Nunc ſi loco x ſubſtituatur ejuſdem valor in æquatione, erit quantitas maximæ ordinatæ $= \frac{9}{64}a^4 + \frac{6}{8}aayy + y^4 - \frac{3}{8}a^4 + aayy = 0$, nempe $y^4 + \frac{14}{8}aayy = \frac{15}{64}a^4$, hoc eſt $yy = \frac{1}{8}aa$.

Quod analytico calculo inventum eſt, ſynthetice demonſtremus. Eſt quidem $AF : FD = AB : BD = FAD : FDA = AK^2 : KR^2$, & $FD : DC = FDC : FCD = ED^2 : EC^2 = ED^2 : AK^2$; eſt enim $EC = AK$; ergo ex æquo perturbate $AF : FC = DE^2 : KR^2$. Sed AF eſt dupla ipſius FC, ergo DE^2 duplum eſt KR^2. Si itaque ponatur ubique $DV^2 = \frac{1}{2}$ *Fig.* 21.

D

Digitized by Google

DE^2, ut fiat Ellipſis CVF, erit $DV = KR$; & VK parallela ipſi AF determinabit in ſubtenſa AB punctum K ad curvam, de qua nunc agitur. Ex quo patet maximam ordinatam ibi reperiri, ubi KV pertingit ad punctum O dimidium Ellipſis, adeoque $KR^2 = \frac{1}{2} QN^2$. Sed $\frac{1}{2} QN^2 = \frac{1}{8} a^2$, ergo patet propoſitum. Denique cum ſit $AR^2 + RK^2 = CD^2 + DE^2 = CD^2 + 2RK^2$, erit $AR^2 = CD^2 + RK^2 = CD^2 + DV^2 = CV^2$; poſita ergo abſciſſa AR, cui maxima reſpondet ordinata, erit $AR^2 = CO^2$. Sed $CO^2 = CN^2 + NO^2 = \frac{1}{4} aa + \frac{1}{8} aa = \frac{3}{8} aa$, ergo $xx = \frac{3}{8} aa = \frac{3}{2}$ quadrati radii CN, vel NQ; unde patet propoſitum.

PROBL. XIII.

158. Propoſitum numerum ita in duas partes dividere, ut factum ex iiſdem ſit omnium maximum.

Propoſitus numerus ſit $= a$, altera ex ejuſdem partibus ſit $= y$, erit altera $= a - y$ & factum ex iiſdem $= ay - yy$, quare, ſumtis differentiis, erit $ady - 2ydy = 0$, & $y = \frac{1}{2} a$.

PROBL. XLIII.

159. Invenire fractionem, cujus quadratum excedat cubum quantitate omnium maxima.

Ponatur quæſita fractio $= x$, erit ergo ejuſdem quadratum $= xx$, & cubus $= x^3$, quare exceſſus quantitatis xx ſupra cubum x^3 exprimetur per $xx - x^3$, hinc $2xdx - 3xxdx = 0$, & $x = \frac{2}{3}$.

PRO-

PROBL. XLIV.

160. Dato ſemicirculo AGVB invenire aggregatum chordæ BG, & ordinatæ GD omnium maximum. *Fig. 36.*

Sit circuli diameter $AB = a$, $BD = x$, erit $DG = \sqrt{(ax - xx)}$, & $BG = \sqrt{ax}$, quare quæſitum aggregatum erit $= \sqrt{[ax - xx]} + \sqrt{ax} = y$. Sumantur differentiæ, erit $\frac{adx - 2xdx}{2\sqrt{(ax - xx)}} + \frac{adx}{2\sqrt{ax}} =$
$\frac{(adx - 2xdx)2\sqrt{ax} + 2adx\sqrt{(ax - xx)}}{2\sqrt{[ax - xx]}2\sqrt{ax}} = dy = 0$;
hinc $a\sqrt{(ax - xx)} = (2x - a)\sqrt{ax}$, & $a^3x - aaxx = 4ax^3 - 4aaxx + a^3x$, nempe $x = \frac{3}{4}a$.

PROBL. XLV.

161. Invenire maximum rectang. ex ſinu complementi GD in GE tangentem ſemiarcus GV. *Fig. 36.*

Sit circuli radius $= a$, $VF = x$, erit $FG = \sqrt{[2ax - xx]}$ & $GD = a - x$. Eſt autem $GF : GC = VF : EG$, $\sqrt{(2ax - xx)} : a = x : \frac{ax}{\sqrt{(2ax - xx)}} = EG$, quo ducto in $GD = FC = a - x$, fiet $\frac{aax - axx}{\sqrt{[2ax - x^2]}}$, ſumtiſque differentiis, & neceſſariis inſtitutis operationibus, erit $\frac{a^3xdx + ax^3dx - 3aaxxdx}{[2ax - xx]^{\frac{3}{2}}}$
$= 0$; Quare $aa = 3ax - xx$, nempe $3a - x : a = a : x$. Fiat $FC = a - x = z$, erit $aa = zz + az = [a + z]z = [VC + FC]FC$. Sed ob ang. rectum tangentis eſt $aa = HCF$, ergo $(VC + FC)FC = HCF$, adeoque $VC + CF = HC$, & $HF = VC$, aut GC, vel $HV = FC$, Quare HC ſecatur extrema, ac media ratione in F. Eſt enim $HF : FC = HF^2 : FG^2 = GC^2 : GF^2 = GH^2 : HF^2 = CH : HF$.

PRO-

PROBL. XLVI.

Fig. 37. 162. Datam rectam A D utcumque ſectam in C iterum ſecare in B, ut aggregatum quadratorum A B, B C, B D ſit omnium minimum.

Eſto $AB = x$, $AC = a$, $CD = b$, erit $BC = a - x$, $BD = a + b - x$; hinc $AB^2 + BC^2 + BD^2 = 3xx - 4ax - 2bx + 2aa + 2ab + bb$, & $6xdx - 4adx - 2bdx = 0$, quare $3x = 2a + b$, & $x = \frac{2a+b}{3} = \frac{1}{3}a + \frac{a+b}{3}$; unde $AB = \frac{1}{3}$ ſummæ A C, A B, & punctum B eſt centrum gravitatis trium punctorum A, C, D.

PROBL. XLVII.

Fig. 37. 163. Datam rectam A D ita ſecare in B, ut productum ex AB^2 in BD^3 ſit omnium aliorum ſimilium productorum maximum.

Ponatur quæſita $AB = x$, & data $AD = a$, erit $AB^2 = xx$, & $BD^3 = a^3 - 3aax + 3axx - x^3$, & AB^2 in $BD^3 = a^3xx - 3aax^3 + 3ax^4 - x^5$, quod quidem ut æquetur maximo, ponenda eſt ejuſdem differentia $= 0$; Erit itaque $2a^3xdx - 9aaxxdx + 12ax^3dx - 5x^4dx = 0$, & $x^3 - \frac{12}{5}axx + \frac{9}{5}aax - \frac{2}{5}a^3 = 0$, cujus æquationis radices ſunt $x - a = 0$, $x - a = 0$, $x - \frac{2}{5}a = 0$, ſive $x = a$, $x = a$, $x = \frac{2}{5}a$, quæ eſt vera radix ad problematis ſolutionem perducens.

PRO-

PROBL. XLVIII.

164. Determinare punctum A in data peripheria circuli ECB, ex quo ſi ad data puncta F, H inclinentur rectæ AF, AH, angulus FAH ſit omnium poſſibilium maximus, aut minimus; oportet autem data puncta, vel ambo ſimul extra, vel ambo intra datam peripheriam contineri. *Fig. 38.*

Ducatur per datum punctum H quævis recta HB ſecans circulum in I, & juncta FH, fiat ut $FH : HB = HI : HG$; erit ergo rectang. $FHG = BHI$; Ducatur nunc ex D tangens circuli GCD, & juncta HC ſecans circulum in A, connectatur FA, quæ circulo in E occurrat, & jungatur EC; Dico factum.

Quoniam enim rectang. $AHC = BHI = FHG$, ergo per puncta A, C, G, F tranſit circulus; quare ang. $HGC = CAE = ECG$; adeoque EC eſt parallela FG; proindeque triang. CAE, HAF ſunt ſimilia, ergo circulus triang. HAF circumſcriptus tanget circulum circumſcriptum CAE, adeoque ang. HAF eſt maximus, aut minimus pro ut ad remotiorem, vel propiorem circumferentiæ partem extenditur; Ubi vero utrumque punctum H, F eſt intra circulum, ſemper ang. eſt maximus eorum, qui ad eaſdem partes fieri poſſunt, ad portionem tamen minorem ſit major, & ad majorem minor.

PROBL. XLIX.

165. Per datum punctum E ſub dato ang. ABC inclinare rectam EDA, ut ſumma laterum reſectorum BA, BD ſit omnium minima. *Fig. 39.*

Ducatur EF parallela ipſi BC, & ſit $EF = a$, $FB = b$, $FA = x$, ergo $AB = x + b$. Et quoniam triang. AFE, ABD ſunt ſimilia, erit $AF : FE = AB : BD$, $x : a = x + b : a + \frac{ab}{x} = BD$; hinc ſumma laterum $= a + \frac{ab}{x} + x + b$, ejuſque differentia $= -$

 ab

$\frac{abdx}{xx}+dx=0$; Quare $abdx=xxdx$, & $ab=xx$, adeoque x media proportionalis inter a, & b, ſive inter EF, FB.

Si quæratur minima ſumma quadratorum AB, BD, erit hæc $= aa+\frac{2aab}{x}+\frac{aabb}{xx}+xx+2bx+bb$, cujus differentia $= -\frac{2aabdx}{xx}-\frac{2aabbxdx}{x^4}+2xdx$ $+2bdx=0$, & inſtituta multiplicatione per x^3, & diviſione per $2dx$ fiet $x^4+bx^3-aabx-aabb$ $=0$, & tandem dividendo per $x+b$ fiet $x^3=aab$; Quare $x:b=aa:xx$, & x prima duarum mediarum continue proportionalium inter a, & b.

Non abſimili ratione invenitur minima ſumma cuborum, & generatim minima ſumma quarumlibet poteſtatum eorundem laterum.

PROBL. L.

Fig. 40. 166. Ex punctis A, B a recta FF inæqualiter diſtantibus rectas AC, BC ad aliquod ejuſdem punctum C ita inclinare, ut rectang. ACB ſit omnium maximum.

Ducatur ipſi EF normalis AE, quæ ponatur $=a$, eidemque normalis altera BF, quæ ſit $=b$, intercepta vero FF ſit $=c$. Ponatur præterea FC $=x$, erit AC $=\sqrt{(aa+xx)}$, & CB $=\sqrt{(cc-2cx+xx+bb)}$; Quare rectang. ACB erit $=\sqrt{[aacc+ccxx-2aacx-2cx^3+aaxx+x^4+aabb+bbxx]}$, cujus differentia $\frac{ccxdx-aacdx-3cxxdx+aaxdx+2x^3dx+bbxdx}{\sqrt{[aacc+ccxx-2aacx-2cx^3+aaxx+x^4+aabb+bbxx]}}$ poni debet $=0$; hinc $2x^3+ccx+aax+bbx=aac$

+

$+ 3cxx$, & inſtituta diviſione per x, erit $2xx + cc + aa + bb = \frac{aac}{x} + 3cx$, & ablato utrinque $xx + 2cx + aa$, habebitur $xx - 2cx + cc + bb = \frac{caa}{x} - xx + cx - aa = \frac{caa - x^3 + cxx - aax}{x}$, hoc eſt $x : c - x = aa + xx : bb + cc - 2cx + xx$, nempe $EC : CF = AC^2 : CB^2$, vel ducta CD parallela ipſi BF, $AD : DB = AC^2 : CB^2$.

Quod autem rectang. ACB ſit minimum, ſic oſtendi poteſt; quoties nempe fuerit $EC : CF = AC^2 : CB^2$. Ponatur $AE = a$, $BF = b$, $EC = x$, $CF = y$ erit $AC = \sqrt{(aa + xx)}$, & $BC = \sqrt{(bb + yy)}$, erit ergo $\sqrt{(aabb + aayy + bbxx + xxyy)} = ACB$, cujus differentia $= \frac{aaydy + bbxdx + yyxdx + xxydy}{\sqrt{(aabb + aayy + bbxx + xxyy)}} = 0$; Eſt autem $EF = c$, ergo $c - x = y$, proindeque $dy = -dx$; Quare $aaydx + xxydx = bbxdx + yyxdx$, vel $-aaydx + bbxdx + yyxdx - xxydx = 0$, unde $(aa + xx)\, y = [bb + yy]\, x$, quare $x : y = aa + xx : bb + yy$, nempe $EC : CF$, vel $AD : DB = AC^2 : CB^2$.

COROLL.

167. Unde conſtat, quod ſi in Caſſinoide GCH ex ejus focis A, B ducantur ad punctum quodvis C rectæ AC, BC, & recta AB ita ſecetur in D, ut ſit $AD : DB = AC^2 : CB^2$, ducta ECF normalis ad CD jungentem puncta C, D, erit tangens curvæ in C.

PROBL. LI.

168. In cono ſcaleno ABED invenire maximum triang. ex ejuſdem coni ſectione efformatum traducto per verticem plano.

Fig. 41. Eſto triang. ex AG tanquam ſua altitudine, & baſi dupla GH normali diametro BD baſis dati coni, cujus axis ſit $AC = b$; Ducatur perpendicularis AK ex vertice in planum baſis, cujus diſtantia a centro baſis C, nempe KC ſit $= c$. Ponatur præterea $BD = 2a$, & $CG = x$, erit $GH = \surd(aa - xx)$, quæ ſi ducatur in $AG = \surd(bb + xx + 2cx)$ emerget

$$\surd(aabb + aaxx + 2aacx - bbxx - x^4 - 2cx^3),$$

qui eſt valor quæſiti triang. Sumatur quantitatis hujuſce differentia, & ponatur $= 0$, habebitur æquatio

$$2aaxdx + 2aacdx = 2bbxdx + 4x^3dx + 6cxxdx,$$

adeoque $x^3 + \frac{2}{3}cxx + \frac{bb}{2}x - \frac{aa}{2}x - \frac{aac}{2} = 0$, quæ eſt æquatio cubica unicam tantum veram habens radicem.

Ut autem hæc reſolvatur æquatio, ad axem AKM per punctum K deſcribatur parabola KNO, cujus parameter $= KC = c$, ſumtaque $KP = \frac{3}{8}c$ ducatur per P recta PQ axi AKM parallela, & parabolæ occurrens in N, & ponatur $NQ = \frac{25c}{32} - \frac{b^2 + a^2}{4c}$, eidemque ſtatuatur normalis $QR = \frac{3099c}{8192} - \frac{3bb - aa}{16c}$; tum centro R, & radio RN deſcribatur circulus ONS ſecans parabolam in O, unde ducatur ad PQ normalis OT, erit hæc $= CG = x$, quæ quærebatur.

PRO-

169. Circulo AEBH parabolam cujuſvis generis FED omnium poſſibilium minimam circumſcribere, in qua nempe $GE^n : BD^n = FG : FB$. *Fig. 42.*

Quoniam $GT = \frac{GE^2}{GC}$, & $GF = \frac{GE^2}{nGC}$, & $BF = \frac{GE^2}{nGC} + GB = \frac{GE^2 + nBGC}{nGC}$, vel (propter $AG + GC = BC$, adeoque $AG + 2GC = GB$, ſive $AGB + 2BGC = BG^2$, vel etiam $BG^2 - 2BGC = AGB = GE^2$) eadem $BF = \frac{BG^2 + [n-2]BGC}{nGC}$.

Quare cum ſit $FG : FB = GE^n : BD^n$, erit $\frac{GE^2}{nGC} : BG^2 + [n-2]BGC = GE^n : [BG^2 + (n-2)BGC]\, GE^{n-2}$, vel etiam $= [GE^2 + NBG]GE^{n-2}$, adeoque $BD = [BG^2 + (n-2)BGC]^{\frac{1}{n}}\, GE^{\frac{n-2}{n}}$, ſive $= [GE^2 + nBGC]^{\frac{1}{n}} GE^{\frac{n-2}{n}}$, & $FBD = \frac{[BG^2 + (n-2)\, BGC]^{\frac{1+n}{n}}\, EG^{\frac{n-2}{n}}}{nGC}$, vel $= \frac{(GE^2 + nBGC)^{\frac{1+n}{n}}\, GE^{\frac{n-2}{n}}}{nGC}$. Eſt autem $n+1 : n$ = rectang. ad parabolam, ut alibi oſtendetur, ergo parabola $FEDE = \frac{[BG^2 + [n-2]BGC]^{\frac{1+n}{n}}\, GE^{\frac{n-2}{n}}}{[1+n]GC}$.

Digitized by Google

ſive $= \frac{(GE^2 + nBGC)^{\frac{1+n}{n}} GE^{\frac{n-2}{n}}}{(1+n)GC}$. Jam vero ponatur $CB = a$, $CG = x$, erit area parabolica circumſcripta circulo generatim accepta $= [aa - xx + nax + nx^2]^{\frac{1+n}{n}} (aa - xx)^{\frac{n-2}{2n}}$, cujus differentia, ſi conſtans denominator $1 + n$ negligatur, erit $\frac{(1+n)}{n} \times$

$(aa + nax + (n-1)x^2)^{\frac{1}{n}} (aa - xx)^{\frac{n-2}{n}} [nadx + 2nxdx - 2xdx] - \frac{[n+2]}{n} x (aa - xx)^{\frac{-2-n}{2n}}$

$(aa + nax + [n-1]x^2)^{\frac{1+n}{n}} xdx - [aa + nax + [n-1]xx]^{\frac{1+n}{n}} (aa - xx)^{\frac{n-2}{2n}} dx = 0$, diviſis omnibus per xx, quo communi diviſore neglecto, & diviſis omnibus per $(aa - xx)^{\frac{-2-n}{2n}} [aa + nax + (n-1)xx]^{\frac{1}{n}} dx$, ceteriſque neceſſariis inſtitutis operationibus, habebitur $a^4 - a^3x - (n-1)aaxx + [n-1]ax^3 + (2n-2)x^4 = 0$.

Hæc autem generalis æquatio ſi dividatur per $x - \frac{a}{2}$, præbet $x^3 + axx - aax - \frac{a^3}{n-1} = 0$, quæ rurſus

ſus dividi poteſt per $x + a$, ut remaneat $xx - \frac{aa}{n-1} = 0$, adeoque $x = \frac{a}{\sqrt{(n-1)}}$; Quare $\sqrt{(n-1)} : 1 = a : x = BC : GC$, & componendo $\sqrt{(n-1)} + 1 : 1 = a + x : x = BG : GC$. Sed $BG : GC = TG : GA$ ſinum verſum ſemicirculi, ergo $\sqrt{(n-1)} + 1 : 1 = TG : GA$; eſt autem $n - 1 > \sqrt{(n-1)}$ quare $n = n - 1 + 1 > \sqrt{[n-1]} + 1$, ergo ratio $n : 1 > \sqrt{[n-1]} + 1 : 1$. Sed $n : 1 = GT : GF$ abſciſſam parabolæ, quare $GT : GF > GT : GA$, proindeque $GF < GA$, & apex parabolæ cadit infra verticem ſemicirculi.

Sit nunc circuli diameter AB partium 48, AG partium 13, proindeque GB partium 35, & rectang. $BGA = 425 = GE^2$, & $GE^4 = 180625$: Et quoniam $CG = 11$, fiet $11 : 35 = GC : GB = GA : GT = 13 \cdot \frac{455}{11}$, cujus quarta pars $\frac{455}{44} = FG$, itaque $\frac{455}{44} : AD = \frac{455}{44} + 35 = \frac{1995}{44} = 180625 : \frac{360346775}{455} = 792168 + \frac{331}{455}$, proindeque BD major eſt quam $\sqrt[4]{792168}$ major quam 29, cujus biquadratum eſſet 707281, & rectang. FBD majus quam $\frac{1995}{44} \cdot 24$ ſcilicet majus quam $\frac{56855}{44}$ majus quam 1292, eadem vero BD minor eſt quam $\sqrt[4]{792169}$, adeoque minor quam 30, cujus biquadratum eſſet 810000, unde FBD minus quam $\frac{59850}{44}$ ſeu quam $1360\frac{5}{22}$.

Si vero iiſdem

Si vero iiſdem poſitis ſumatur AK partium 4, & fiant quæ ſupra, invenietur rectang. VLB minus quam $1339\frac{4}{5}$, quod minus eſt rectang. FBD.

Rurſus ſi adſumatur AK partium 3, erit rectang. VLB minus quam 1305., adhuc minus præcedenti; Quare & parabola VHLB minor eſt quam parabola AEDB.

COROLL.

170. Unde conſtat parabolam illam, quam doctiſſimus CÆLESTINUS ROLLIUS in epiſtola, quæ præfigitur geometricæ exercitationi Cl. LAURENTII LORENZINI, putat minimam, revera eſſe maximam.

PROBL. LIII.

Fig. 43. 171. Per datum punctum D ſub ang. dato ACB ducere rectam ADB, quæ minima ſit omnium per idem punctum D ſub eodem ang. ductarum.

Ponatur factum quod quæritur, & recta AB ſit quæſita. Ex puncto D ducantur rectæ DF, DE parallelæ lateribus CB, CA, & DG normalis ſupra AC. Ponatur $AF = x$, $DF = a$, $DE = FC = b$, $FG = c$, erit ergo $AG = x \mp c$, $DG = \sqrt{(aa - cc)}$ eſt $AD = \sqrt{(xx \mp 2cx + aa)}$. Eſt autem ob ſimilit. triang. ADF, ABC, $AF : FD = AC : CB$, $x : a = x + b : \frac{ax + ab}{x} = CB$, & $FD : DA = CB : BA$, $a : \sqrt{[xx \mp 2cx + aa]} = \frac{ax + ab}{x} : \frac{[x + b]}{x} \sqrt{[xx \mp 2cx + aa]} = BA$; quæ ut ſit minima, ſumi debet ejuſdem differentia, & fieri $= 0$; erit igitur $x^3 dx \mp cxxdx \mp bcxdx - aabdx = 0$, unde habetur $x^3 \mp cxx \mp bcx - aab = 0$.

COROLL.

COROLL. I.

172. Si datum punctum D foret in recta C H bifariam ſecante datum angulum A C B, tunc a eſſet $= b$, & ſuperior æquatio verteretur in $x^3 \mp cxx \pm acx - a^3 = 0$, quæ dividi poteſt per $x - a = 0$, unde $x = a$, triang. A C B eſſet iſoſcelion, & A D = D B.

COROLL. II.

173. Si triang. A C B ponatur iſoſcelion, & punctum datum D non ſit in recta C H fieri nunquam poterit ut baſis A B ſit minima omnium per idem punctum D ſub eodem ang. ductarum. Hoc ſi quidem caſu foret D F = F A, ſive $a = x$. Poſita igitur in prima æquatione quantitate a loco x, & inſtituta diviſione per $aa \mp ac$, eſſet $a = b$, major minori, quod eſt abſurdum. Falſum ergo baſim triang. iſoſcelis eſſe minimam rectarum, quæ duci poſſunt per quodvis ejus punctum, & lateribus intercipi, ut nonnulli putaverunt.

COROLL. III.

174. Si datus ang. C ſit rectus, tunc $c = 0$, unde prior æquatio ſit $x^3 - aab = 0$, & $x = \sqrt[3]{aab}$; Eſt autem A F : F D = D E : E B, $x : a = b : \frac{ab}{x} = \frac{ab}{\sqrt[3]{aab}} = \sqrt[3]{abb}$; Quare in hoc caſu A F eſt prima, & E B ſecunda ex duabus mediis proportionalibus inter D F, & D E.

CAP. III.

De uſu Calculi differentiarum diverſi ordinis in inveniendo centro generali Linearum geometricarum.

DEFINITIO.

175. Centrum generale curvarum dicitur punctum A in earundem plano ſitum, quo in puncto rectæ omnes per ipſum tranſeuntes ita ſecantur, ut portiones inter punctum illud, & diverſos curvæ ramos interceptæ ſint æquales portionibus inter idem punctum, & ramos ad alteram oppoſitamque regionem vergentes interjectis: nempe ſi AB = AC, punctum A dicetur centrum generale.

Fig 44

COROLL.

176. Hinc patet, quod ſi ponamus, punctum, in quod origo coordinatarum transferri poteſt, quoties addantur reſpective tum abſciſſæ x, tum ordinatæ y quantitates indeterminatæ m, & n, in centrum generale evaſiſſe, valores ordinatarum poſitivi æquales erunt ſingillatim valoribus earundem negativis, hoc eſt æquantur zero coefficientes omnes parium terminorum in æquatione permutata, ubi ordinatæ ipſæ reſpondent abſciſſæ, quæ ſit = o.

LEMM.

177. Æquationem curvæ propoſitam in aliam commutare, in qua ordinatæ ſub ang. quocumque ad axem inclinentur.

Exprimantur novæ coordinatæ per z, & v, & loco x, & y ſufficiantur reſpective valores $z+qv$, & pv, confectumque erit problema, ut demonſtrant Analyſtæ, & nos oſtendemus in algebra Finitorum, quam brevi proferemus in lucem.

CO-

COROLL. I.

178. Hinc ut origo indeterminatarum in quodvis punctum transferatur, loco x ſufficienda erit quantitas $m + z + qv$, & loco y quantitas $n + pv$.

COROLL. II.

179. Ergo, ſi punctum, in quod origo coordinatarum transfertur, ſtatuatur eſſe centrum generale, indeterminata z efficienda erit $= 0$, ſive ſubſtituenda tantum erit quantitas $m + qv$ loco x, & $n + pv$ loco y.

COROLL. III.

180. Quod ſi in æquatione propoſita ponamus quantitates $x + dx$, & $y + dy$ in locum x, & y ſufſectas eſſe, nempe m per x, qv per dx, n per y, & pv per dy repræſentari, facilius innoteſcent termini poſtremæ hujus æquationis permutatæ, quæ ex ſubſtitutione quantitatum $m + qv$ & $n + pv$ loco x, & y in propoſita æquatione prodiiſſet. Et quidem ſi tollantur ex variis differentiis datæ æquationis differentiæ illæ, quæ primam dimenſionem excedunt, & ſecunda dividatur per 2, tertia per 2, & per 3, quarta per 2, 3, 4, atque ita porro, hujuſmodi variæ differentiæ præbebunt inverſo quidem ordine diverſa permutatæ æquationis membra ex ſubſtitutione $x + dx$, & $y + dy$ prodeuntis, ut Cl. Viri BERNOULLIUS, & SAURINUS demonſtrarunt. *

**Mem. de l'Acad. Roy. des Sciences an. 1711., & 1716.*

PROBL. LIV.

181. Ex data æquatione ad curvam determinare I. Utrum hæc centro generali donetur; II. Quibus poſitis conditionibus curva ipſa hujuſmodi centrum acquirat; III. Quo in puncto plani centrum ipſum ſtatui poſſit.

Digitized by Google

Sumantur ordine quotquot ex æquatione propoſita erui poſſunt differentiæ, poſtrema tantum excepta, & tollantur differentiæ, quarum dimenſiones excedunt primam. Mox in ſingulis quibuſque differentiis imparis dimenſionis, ſi æquatio propoſita fuerit paris dimenſionis, vel in differentiis paris dimenſionis, ſi æquatio fuerit imparis, habita ratione ipſiuſmet æquationis, quæ haberi poſſet ut differentia dimenſionis zero, ſi fiat ſingillatim quodlibet membrum $= 0$, eruentur valores ipſius x, & y, quorum ope tum centrum generale, tum conditiones quoad coefficientes datæ æquationis, ex quibus pendet ejuſdem centri exiſtentia, facile poterunt determinari.

PROBL. LV.

182. Data æquatione ad circulum, ejuſdem centrum generale invenire.

Sit igitur æquatio $yy = 2ax - xx$; quæ eſt paris dimenſionis; patet unam tantum imparem differentiam, primam ſcilicet ex hac æquatione erui poſſe; eſt nempe $2ydy = 2adx - 2xdx$, ſive $2ydy - 2adx + 2xdx = 0$; fiat nunc ſeorſum quodlibet membrum hujuſce differentiæ $= 0$, erit $y = 0$, & $x = a$, nullæ vero hic habentur conditiones; Curva igitur propoſitæ æquationis donatur centro generali, quod in axe reperitur, & diſtat ab origine quantitate a.

SCHOL.

183. Si origo abſciſſarum ſumatur in puncto medio diametri, quoniam tunc æquatio eſt $yy = aa - xx$, erit $2ydy = -2xdx$ ergo $y = 0$, & $x = 0$, adeoque centrum generale, quod nullis hic adſigitur conditionibus, ſitum eſt in origine abſciſſarum.

PROBL. LVI.

184. Invenire centrum generale ellipſeos.

Quoniam

Quoniam in æquatione ad ellipſim $ayy = abx - bxx$, quæ eſt paris dimenſionis, & ex qua nonniſi unica differenria impar ſumi poteſt, nempe prima, eſt $2aydy = abdx - 2bxdx$, ſi fiat ſingillatim quodlibet membrum $= 0$, erit $y = 0$, & $x = \frac{1}{2}a$, nullæ autem hic adſunt conditiones, ergo centrum generale reperitur in axe ellipſeos ad diſtantiam $\frac{1}{2}a$ ab origine.

Idem invenietur in æquatione ad hyperbolam $ayy = abx + bxx$.

SCHOL.

185. Si fuerit æquatio ad ellipſim $2ayy = aab - bxx$, poſita origine abſciſſarum in puncto bifariam ſecante diametrum, erit $4aydy = -2bxdx$, adeoque $y = 0$, $x = 0$, $b = 0$, $a = 0$, ergo nullæ hic adſunt conditiones, propterea que centrum generale reperitur in origine abſciſſarum.

PROBL. LVII.

186. Data æquatione $yy + xx - 2by - 2ax - \frac{2bgx}{c} - \frac{2gxy}{c} + \frac{ggxx}{cc} + bb = 0$ ad circulum, in qua adeſt planum xy, invenire ejuſdem centrum generale.

Eſt quidem una differentia impar, quæ ex data æquatione erui poteſt $2ydy + 2xdx - 2bdy - 2adx - \frac{2bgdx}{c} - \frac{2gxdy}{c} - \frac{2gydx}{c} + \frac{2ggxdx}{cc} = 0$, ſive $2ydy - 2bdy - \frac{2gxdy}{c} = \frac{2gydx}{c} - \frac{2ggxdx}{c} + \frac{2bgdx}{c} + 2adx - 2xdx$; quare $y - b - \frac{gx}{c}$

$= o$, $\frac{gy}{c} - \frac{ggx}{cc} + \frac{bg}{c} + a - x = o$, erit ergo I. $\frac{cy - bc}{g} = x$, II. $\frac{cgy + bcg + acc}{cc + gg} = x$; hinc $\frac{cy - bc}{c} = \frac{cgy + bcg + acc}{cc + gg}$, hoc eſt $cggy + cccy - bcgg - bccc = ccgy + bccg + accc$, & $y = \frac{bcg + acc + bgg + bcc}{gg + cc - cg}$; At erat $y - b - \frac{gx}{c} = o$, & $\frac{gy}{c} - \frac{ggx}{cc} + \frac{bg}{c} + a - x = o$, quare erit III. $y = b + \frac{gx}{c}$, IV. $y = \frac{gx}{c} - b - \frac{ac}{g} + \frac{cx}{g}$, ac propterea $b + \frac{gx}{c} = \frac{gx}{c} - b - \frac{ac}{g} + \frac{cx}{g}$, unde $2b + \frac{ac}{g} = \frac{cx}{g}$, & $x = \frac{2bg + ac}{c}$. Ergo centrum generale determinabitur, ſi adſignentur indeterminatis quantitatibus x, & y valores adinventi.

Eodem plane modo procedendum in Ellipſi, & hyperbola, ſi in earundem æquatione adeſſet planum xy.

PROBL. LVIII.

187. Invenire centrum generale in curva æquationis $y^3 - x^3 + axx - bbx = o$.

Eſt profecto prima differentia hujuſce æquationis $3yydy - 3xxdx + 2axdx - bbdx = o$, ſive $3yydy = 3xxdx - 2axdx + bbdx$; ſecunda vero ejuſdem differentia (neglectis ſecundis differentiis) eſt $6ydy^2 = 6xdx^2 - 2adx^2$; Tertia vero differentia hic minime conſideratur, eſſet quippe poſtrema,

quæ

quæ ex data æquatione erui poſſet. Fiant nunc duo membra ſecundæ differentiæ ſingillatim $= 0$, erit $y = 0$, & $x = \frac{1}{3} a$; ſufficiantur hi valores in æquatione propoſita, fiet I. $a = 0$, adeoque $x = 0$; II. poſito $\frac{1}{3} a$ loco x, fiet $xxx - axx + bbx = \frac{1}{27} aa - \frac{1}{9} aa + \frac{1}{3} bb$, adeoque $\frac{1}{3} bb = \frac{3}{27} aa - \frac{1}{27} aa = \frac{2}{27} aa$, unde tandem $9bb = 2aa$; Ergo ubi $a = 0$ centrum generale erit in origine curvæ, ubi vero $9bb = 2aa$, centrum generale ſitum erit in axe, & diſtabit ab origine quantitate $\frac{a}{3}$.

PROBL. LIX.

188. Invenire centrum generale in hyperbolis redundantibus tertii ordinis linearum, quarum natura exprimi poteſt per æquationem Newtonianam $xyy + ey = axxx + bxx + cx + g$.

Sumatur propoſitæ æquationis prima, & ſecunda differentia, erit I. $2xydy + edy = 3axxdx + 2bxdx - yydx + cdx$; II. neglectis differentis ſecundis, habebitur $2xdy^2 = 6axdx^2 + 2bdx^2 - 2ydydx$, vel $2xdy^2 + 2ydydx = 6axdx^2 + 2bdx^2$; Differentia tertia hic omittitur, utpote poſtrema.

Jam vero æquationes, quæ erui poſſunt ex differentia ſecunda ſunt I. $2x = 0$, vel $x = 0$, II. $2y = 0$, vel $y = 0$, III. $6ax + 2b = 0$, æquatio vero, quæ haberi poteſt ex differentia, cujus dimenſio ſit zero, eſt $xyy + ey = axxx + bxx + cx + g$. In

duabus

duabus hiſce poſtremis æquationibus ſubſtituantur valores ipſius x, & y ſupra inventi, nempe zero loco utriuſque, habebitur $2b = 0$, vel $b = 0$, & $g = 0$. Ergo, ſi curvæ, quas exhibet æquatio propoſita, centro generali donentur, hujuſmodi centrum ſitum eſſe debet in origine; Curvæ autem ipſæ tunc ſolum centro generali donabuntur, cum coefficientes b, & g deſint in æquatione, ut patet.

PROBL. LX.

189. Invenire centrum generale in curva, ad quam $aayy = 2aaax - 5aaxx + 4axxx - x^4$.

Accipiantur prima, & tertia differentia propoſitæ æquationis, erit I. $2aaydy = 2aaadx - 10aaxdx + 12axxdx - 4xxxdx$, III. vero $0 = 24ad\,x^3 - 24x\,dx^3$; quarta autem negligitur, utpote quæ foret poſtrema.

Erit igitur $24a\,dx^3 = 24x\,dx^3$, & $x = a$. Centrum igitur generale ſitum eſt in axe curvæ, & diſtat ab ejuſdem origine quantitate a.

PROBL. LXI.

190. Invenire centrum generale in Caſſinoide, ſumta ejus origine in puncto medio lineæ jungentes focos.

Quoniam æquatio Caſſinoidis eſt $y^4 + 2xxyy + 2bbyy = 2bbxx - x^4 + 2aabb - a^4$, erit ejuſdem prima differentia $4yyydy + 4xxydy + 4yyxdx + 4bbydy = 4bbxdx - 4xxxdx$, tertia vero $2ydy^3 + 24xdxdy^2 + 24ydydx^2 = -24xdx^3$, quare $y = 0$, & $x = 0$, ergo hujuſmodi curva habet neceſſario centrum generale, quod ſitum eſt in ipſa origine.

PROBL.

PROBL. LXII.

191. Invenire centrum generale in curva æquationis $9aay^4 = 4x^6 + 12aax^4 + 12a^4 xx + 4a^6$.

Eſt differentia prima propoſitæ æquationis $36aayyydy = 24x^5 dx + 48aaxxxdx + 48a^4 xdx$, tertia vero $216aaydy^3 = 288aaxdx^3 + 480x^3 dx^3$, quinta demum, omiſſa ſexta differentia, utpote poſtrema, eſt $o = 2880xdx^5$, hinc $x = o$, & pariter $y = o$, proindeque exiſtentia centri generalis in hujuſmodi curva nullis adſigitur conditionibus; Eſt autem $x = o$, ergo centrum generale reperitur in origine abſciſſarum.

CAP. IV.

De uſu Calculi differentiarum diverſi ordinis in puncto Flexus contrarii Curvarum determinando.

DEFINITIO.

192. Punctum *Flexus contrarii* in curva dicitur punctum, quod terminus eſt partem concavam a convexa diſtinguens. Ita ſi fuerit curva ABCD, quæ primo concavitatem axi AF juxta ductum ABC, deinde juxta flexum CD convexitatem eidem obvertat, vel primo convexitatem axi AF juxta ductum ABC, mox concavitatem juxta flexum CD eidem axi opponat, punctum C dicitur punctum flexus contrarii, quo nomine punctum illud appellatur, tum ſi ordinatæ ſint inter ſe parallelæ, tum etiam ſi ab aliquo fixo puncto, quod *polus* dicitur, procedant. *Fig. 46.*

THEOREMA I.

193. Quantitas dy^2 eſt infinite parva relate ad dy, & ordinis ad hanc ſuperioris.

Eſt enim dy^2 tertia proportionalis ad 1, & dy; Eſt nempe $1 : dy = dy : \frac{dy^2}{1} = dy^2$. Sed dy eſt infinite parva relate ad 1, ergo dy^2 eſt infinite parva relate ad dy; eſt autem dy infiniteſima primi ordinis, ergo dy^2 eſt infiniteſima ſecundi; Idem dicas de quantitatibus dx^2, dv^2 &c. relate ad dx, dv &c.

THEOREMA II.

194. Differentiæ ſecundæ ordinatarum continue creſcentium in curva concavitatem axi obvertente ſunt negativæ.

Quoniam creſcentibus abſciſſis creſcunt & ordinatæ, earundem differentiæ erunt poſitivæ (§. 19. *n.* 1.) Quoniam vero ipſarum ordinatarum differentiæ minuuntur (§. 75.) differentiæ hujuſmodi differentiarum, hoc eſt differentiæ ſecundæ ordinatarum erunt negativæ (§. 19. *num.* 2.)

THEOREMA III.

195. Differentiæ ſecundæ ordinatarum ſemper creſcentium in curva convexitatem axi obvertente ſunt poſitivæ.

Creſcentibus ſiquidem abſciſſis creſcunt & ordinatæ, ergo differentiæ hujuſmodi ordinatarum ſunt poſitivæ (§. 19. *num.* 1.). Sed eædem differentiæ jugiter creſcunt (§. 86.), ergo ſunt & ipſæ poſitivæ.

CO-

196. Hinc erui facile poteſt, utrum portio curvæ plures flexus contrarios habentis inter duo hujuſmodi puncta jacens convexa ſit, an concava verſus axem. Si enimvero valor expreſſionis pro ſecunda differentia ordinatæ inter duas reſpondentes radices ſit poſitivus, curva obvertet axi convexitatem, ſi negativus concavitatem, modo integra curva ad unam eandemque plagam ejuſdem axis jaceat. Ita poſita æquatione $y = \frac{180xx}{a} - \frac{110xxx}{aa} + \frac{30x^4}{aaa} - \frac{3x^5}{a^4}$, quoniam ejuſdem differentia ſecunda eſt $= 6aaa - 11aax + 6axx - xxx = (a-x)(2a-x)(3a-x)$, & valor expreſſionis, dum $x < a$, eſt poſitivus, curva ipſa convexa primum erit ad axem. Sed ubi $x > a$, quoniam tunc expreſſio fit negativa, curva fiet concava ad eundem axem, ſicque procedet, quandiu $x = 2a$, deinde vero cum differentia evadat adfirmativa, curva rurſus convexa fiet, donec $x = 3a$, quo prætergreſſo limite curvatura in infinitum tendet ad eandem plagam.

THEOREMA IV.

197. Si exiſtente differentia ſecunda ordinatæ $= 0$, hoc eſt $ddy = 0$; ejuſdem differentia tertia ſit realis, & finita, tunc y tranſit per punctum flexus contrarii. *Fig. 45.*

Hoc ex Cl. MAC. LAURINO ſic demonſtratur. Quoniam puncta flexus contrarii determinantur ponendo $ddy = 0$, vel oo, ut oſtendemus, occurrat recta PM in puncto V tangenti curvam ad punctum F, & pm eidem occurrat in puncto v, tunc $PV = E + \frac{dEx}{dx}$, & $pv = E - \frac{dEx}{dx}$. Sed quando $ddE = 0$,

$$PM = E + \frac{dEx}{dx} * + \frac{dddEx^3}{dx^3} + \&c., \ \& \ pm =$$

$E - \frac{dEx}{dx} * - \frac{dddEx^3}{dx^3} +$ &c. ergo ſi $dddE$ ſit poſitiva, & diſtantiæ AP, Ap ſint minimæ, PM erit $>$ PV, & $pm < pv$, & ſi $dddE$ ſit poſitiva, vel negativa, arcus FM, & Fm erunt ad diverſas plagas tangentis TFt, adeoque punctum F erit punctum flexus contrarii. At ſi $dddE$ pariter evaneſcat, & $ddddE$ ſit valoris realis, PM, & pm erunt ambæ majores, vel minores reſpectivis normalibus PV, & pv a tangente interceptis, nullumque aderit punctum flexus contrarii in F. Generatim ſi ddy, $dddy$, $ddddy$ &c. evaneſcant, ſique numerus harum fluxionum exiſtente impare, fluxio ordinis ſubſequentis habeat valorem realem, & finitum, tunc y tranſit per punctum flexus contrarii. Sed ſi numerus harum fluxionum, quæ evaneſcunt ſit par, nullum erit punctum flexus contrarii, ſed potius duplex flexus infinite parvus Ex. gr. ſit $y = 1 - x^4$, erit $dy = -4xxxdx$, $ddy = -12xxdx^2$, $dddy = -24xdx^3$, $ddddy = -24dx^4$. Si ponatur $ddy = 0$, erit $x = 0$; Sed quoniam tunc $dddy = 0$, & $ddddy$ realis, & finita, y non tranſit per punctum flexus contrarii, ſed revera eſt maximum.

THEOREMA V.

198. Si curva ponatur utrinque progredi ab ordinata PM, vel y, poſitaque $ddy = \infty$, ejuſdem ſigna ſint diverſa, punctum M erit punctum flexus contrarii.

Hoc per ſe patet. Sit ex. gr. $y = 1 - x^{\frac{5}{3}}$, erit $ddy = -\frac{10dx^2}{9\sqrt[3]{x}}$, quæ ſit infinita, ubi $x = 0$, vel

$y = 1$,

$y = 1$, & contrariis ſignis afficitur ad diverſas plagas ipſius y, adeoque ordinata tranſit per punctum flexus contrarii, ubi $x = 0$.

PROBL. LXIII.

199. Invenire formulam, cujus ope determinari poſſit punctum flexus contrarii in curvis, quarum ordinatæ ſunt inter ſe parallelæ.

Poſita dx conſtanti, hoc eſt abſciſſa infinite parva ſibi ſemper æquali, ſumantur differentiæ ſecundæ ordinatarum ad datam curvam, ex quibus fiat ſeries reſpondens ſeriei ordinatarum ad ipſam curvam & per earum extrema traducatur linea, manifeſtum eſt quod in puncto flexus contrarii hujuſmodi differentiæ ex poſitivis fieri debent negativæ, vel contra ex negativis poſitivæ, qui quidem tranſitus fieri debet vel per 0, vel per 00, ut oſtendemus in Algebra Finitorum; quare ſi ſumtis differentiis æquationis ad curvam, poſitaque dx conſtante fiat $ddy = 0$, vel 00, invenietur valor abſciſſæ, cui reſpondet ordinata puncto flexus contrarii occurrens. *

* *Vid. Cl. Fontenellium part. 1. ſect. 10. 11.*

Aliter. Sit aliqua curvæ tangens Bt inter verticem A, & punctum flexus contrarii C, hæc profecto tangens eo altius occurret axi AF producto, quo fuerit punctum B puncto flexus contrarii vicinius. Sumto vero aliquo puncto D poſt immutatam curvaturam, ejus tangens DS inferius deſcendet, nempe punctum S, quod eſt ejuſdem tangentis occurſus cum axe verſus verticem A redibit, quem etiam attingere, imo & tranſgredi poteſt, ſi ducantur tangentes ad alia puncta, quæ ſint ab ipſo puncto C remotiora; Ex quo patet interceptam AT eſſe maximam omnium ſimilium curvæ functionum inter verticem, & occurſum tangentis interceptarum. Ergo in hoc caſu ad determinandum punctum flexus contrarii, quoniam ſubtangens ſemper $= \frac{y\,dx}{dy}$, ejuſdem differentia ab x, nempe $\frac{y\,dx}{dy} - x$ maxima

Fig. 46.

Digitized by Google

maxima ſit oportet, ergo ipſius differentia ſecunda erit $= 0$, proindeque ſumta dx pro conſtante invenietur differentia ſecunda propoſitæ formulæ $= \frac{dy^2\,dx - y\,dx\,ddy - dy^2\,dx}{dy^2} = 0$, hoc eſt $\frac{-y\,dx\,ddy}{dy^2} = 0$, inſtitutaque multiplicatione per dy^2, & diviſione per $-ydx$, fiet differentia ſecunda ordinatæ pertingentis ad punctum flexus contrarii $ddy = 0$.

Quod ſi curva ACD ſit primum ad axem convexa, ducta tangente CT punctum T remotius eſt a vertice A, quam puncta S vel t tangentium DS, vel Bt, & ſubtangens GT eſt $= \frac{y\,dx}{dy}$, adeoque AT $= x - \frac{y\,dx}{dy}$, & cum ipſa AT major eſſe debeat tum ipſa AS, tum ipſa At neceſſe eſt, ut differentia ſecunda ejuſdem AT fiat $= \infty$, quare ſumta dx conſtante cum ſit differentia ipſius $x - \frac{y\,dx}{dy} = \frac{dx\,dy^2 - dx\,dy^2 + y\,dx\,ddy}{dy^2} = \frac{y\,dx\,ddy}{dy^2}$, fiet $ddy = \infty$.

SCHOL.

200. Adnotandum hic eſt hujuſmodi methodum determinandi puncta flexuum contrariorum, quam tradit Illuſtriſsimus HOSPITALIUS in Analyſi infinite parvorum, aliqua æquivocatione laborare. Fatendum enim eſt per eam nullo modo innoteſcere quibus in caſibus altera ex adlatis poſitionibus ſit alteri præferenda, adeoque nonniſi tentando ad id quod quærimus perveniri poſſe. Certum præterea eſt per eandem methodum determinari non modo puncta flexus contrarii, ſed etiam *regreſſus*, vel *cuſpidis*, puncta ſcilicet, in

quibus

quibus curvæ reflectuntur ab ordinatis, ut oſtendi facile poteſt. Quid quod eadem methodus inepta eſt, ut patet, inventioni puncti flexus contrarii, ubi ejuſdem directio eſſe debet axi parallela. Eo ſiquidem caſu cum ſubtangens fiat infinita, eadem ipſa, ſi appoſite loqui velimus, dici non poterit maxima, vel minima.

Verum enim vero cum problemata longe plurima ad puncta flexus contrarii pertinentia tradita methodo conficiantur, & quidem plerumque per formulam $ddy = 0$, punctum namque flexus contrarii, quod idem HOSPITALIUS eruit ab æquatione $y - a = a$

$(x-a)^{\frac{3}{5}}$, ponendo $ddy = \infty$, non eſt punctum ſingulare flexus contrarii, ut ipſe arbitratur, ſed revera punctum triplex, idcirco adlatæ methodo inhærendum nobis erit in enodandis quæſtionibus ad puncta flexus contrarii pertinentibus, cum per eam hujuſmodi quæſtiones nullo negotio reſolvantur, quod ſane per alias regulas, de quibus agemus in tractatu de lineis tertii ordinis, obtineri non poteſt.

PROBL. LXIV.

201. Sumto quovis puncto D in recta AB fiat $AB^2 : AD^2 = BD : DG$, & ſic ſemper quæritur punctum flexus contrarii in curva per puncta G incedente. *Fig 7.*

Sit $AB = a$, $AD = x$, $DG = y$, erit $BD = a - x$, quare habebitur proportio $aa : xx = a - x : y$, unde $aay = axx - xxx$, quæ erit curvæ æquatio; Sumtis igitur differentiis erit $aady = 2axdx - 3xxdx$, poſitaque dx conſtante, $aaddy = 2adx^2 - 6xdx^2$, & $ddy = \frac{2adx^2 - 6xdx^2}{aa}$; Fiat $ddy = 0$, erit $aa = 0$, & $2adx^2 - 6xdx^2 = 0$, hinc $2a = 6x$, & $x = \frac{1}{3}a$.

Digitized by Google

Modo ſufficiatur valor ipſius x in ſuperiore æquatione, erit $y = \frac{2}{27}a$. Quod ſi $\frac{1}{3}a$, & $\frac{2}{27}a$ ad rectum ang. jungantur, determinabitur punctum flexus contrarii, licet curva nondum fuerit deſcripta.

PROBL. LXV.

202. Invenire punctum flexus contrarii in curva æquationis $xyy = 2aaa - aax$.

Eſt quidem $2xydy + yydx = -aadx$, & $dy = -\frac{aaadx}{\sqrt{(2a^3x^3 - a^2x^4)}} = -\frac{aaadx}{z}$, hinc ſumta dx conſtante erit $ddy = \frac{aaadxdz}{zz}$. Sufficiantur loco dz, & zz eorum valores, & erit $ddy = \frac{6a^6xxdx^2 - 4a^5xxxdx^2}{(2a^3x^3 - aax^4)\sqrt{(2a^3x^3 - a^2x^4)}} = 0$, ergo $6a^6xxdx^2 = 4a^5xxxdx^2$, & $x = \frac{3}{2}a$.

Quod ſi valor ipſius x in æquatione data ſufficiatur, erit $y = \sqrt{\frac{1}{3}aa}$.

PROBL. LXVI.

203. Invenire punctum flexus contrarii in curva prop. 4. Quadrat. Circ. Cl. GUIDONIS GRANDI, ad quam curvam $y = \frac{axx}{aa + xx}$.

Eſt dy

Eſt $dy = \frac{2aaaxdx}{[aa+xx]^2} = \frac{2aaaxdx}{z}$, quare ddy $= \frac{2aaaazdx^2 - 2aaaaxdxdz}{zz} =$ $\frac{a^7dx^2 - 2a^5xxdx^2 - 3aaax^4dx^2}{(aa+xx)^4}$, ſi loco z, zz, & dz eorum valores ſufficiantur. Fiat ddy $= 0$, erit $a^7 - 2a^5xx - 3aaax^4 = 0$, ſive $a^4 -$ $2aaxx - 3x^4 = 0$. Inſtituitur diviſio per $aa+xx$, erit $aa - 3xx = 0$, quare $aa = 3xx$, & $x =$ $\sqrt{\frac{1}{3}aa}$, proindeque $y = \frac{1}{4}a$.

PROBL. LXVII.

204. Invenire punctum flexus contrarii in curva æquationis $yy = \frac{aax}{a-x}$.

Erit ergo $2ydy = \frac{aaadx}{(a-x)^2} = \frac{aaadx}{z}$, & ſumta dx conſtante erit $2dy^2 + 2yddy = -\frac{aaadxdz}{zz}$, & $2yddy = -\frac{aaadxdz}{zz} - 2dy^2 = -$ $\frac{aaadxdz - 2zzdy^2}{zz}$, $ddy = -\frac{aaadxdz - 2zzdy^2}{2yzz}$

O =

$$= \frac{2a^4 dx^2 - 2aaaxdx^2 - \frac{a^7 dx^2 + a^6 x dx^2}{2aax}}{2yzz} =$$

o, ſi nempe fiant neceſſariæ ſubſtitutiones; hinc $2a^4 - 2aaax - \frac{a^7 + a^6 x}{2aax} = o$, hoc eſt $4xx - 5ax + aa = o$; Inſtituatur diviſio per $x - a$, fiet $4x - a = o$, & $x = \frac{1}{4} a$.

Aliter. Quoniam $yy = \frac{aax}{a-x}$, vel $ayy - xyy = aax$, ſive $ayy = aax + xyy$, erit $2aydy = aadx + 2xydy + yydx$, poſitaque dx conſtante, erit $2ady^2 + 2ayddy = 2ydxdy + 2xdy^2 + 2xyddy + 2ydydx$, quare $ddy = \frac{4ydydx + 2xdy^2 - 2ady^2}{2ay - 2xy} = o$, ergo $4ydydx + 2xdy^2 - 2ady^2 = o$, ſive $4ydx + 2xdy - 2ady = o$, hinc $2ydx = ady - xdy$, & $\frac{ydx}{dy} = \frac{a-x}{2}$; ſed $\frac{ydx}{dy} = \frac{2ax - 2xx}{a}$ (§. 40.) ergo $\frac{a-x}{2} = \frac{2ax - 2xx}{a}$, ſive $\frac{1}{2} a = 2x$, & $x = \frac{1}{4} a$, prorſus ut ſupra.

Alio modo. Quoniam $ddy = \frac{4ydydx + 2xdy^2 - 2ady^2}{2ay - 2xy} = o$, erit etiam $4ydx + 2xdy - 2ady = o$, poſitoque loco dy ejus valo-

re

Digitized by Google

re $\frac{aadx+yydx}{2ay-2xy}$, erit $4ydx(2x-2a)\frac{[aadx+yydx]}{2ay-2xy}$ $=0$, hoc eſt $2aax-2aaa=6xyy-6ayy$. Loco yy ſuſſiciatur ejus valor $\frac{aax}{a-x}$, erit $2aax-2aaa=$ $\frac{6aaxx-6aaax}{a-x}$, ſive $2aaax-2aaxx-2a^4+$ $2aaax=6aaxx-6aaax$, nempe $4xx-5ax+$ $aa=0$. Inſtituatur diviſio per $x-a$, erit $4x-a=$ 0, & $x=\frac{1}{4}a$, ut ſupra.

PROBL. LXVIII.

205. Invenire punctum flexus contrarii in curva, cujus æquatio $y=\frac{ax}{V(rx-xx)}$.

Fiat $V(rx-xx)=z$, erit $y=\frac{ax}{V(rx-xx)}$ $=\frac{ax}{z}$, quare $dy=\frac{azdx-axdz}{zz}$. Loco z, zz, & dz ponantur eorundem valores erit $dy=\frac{adxV[rx-xx]}{rx-xx}$ $-\frac{ax[rdx-2xdx]}{2[rx-xx]V[rx-xx]}=\frac{arxdx}{2[rx-xx]V[rx-xx]}$. Ponatur nunc $V(vx-xx)=v$, erit $dv=\frac{rdx-2xdx}{2V[vx-xx]}$, & $dy=\frac{arxdx}{2v^3}$; hinc $ddy=\frac{arvdx^2-3arxdxdv}{2v^4}=0$, proindeque $arvdx^2=3arxdxdv$. Reſtituantur in locum v, & dv eorundem valores, erit $ardx^2V[rx-xx]$

 $=$

$= \frac{3arxdx\ (rdx - 2xdx)}{2V(rx - xx)}$. Inſtituatur multiplicatio per $2V(rx - xx)$ erit $2arrx - 2arxx = 3arrx - 6arxx$, ſive $4arxx = arrx$, & $x = \frac{1}{4}r$.

PROBL. LXIX.

206. Invenire punctum flexus contrarii in curva, ad quam $y = V[x + xxx]$.

Erit igitur $dy = \frac{dx + 3xxdx}{2V[x + xxx]} = \frac{v}{z}$, ergo $ddy =$

$\frac{zdv - vdz}{zz} = \frac{6xdx^2\ 2V[x + xxx] —}{4x + 12xxx}$

$\frac{[dx - 3xxdx]\ [4dx + 12x^2\ dx]}{(4x + 12xxx)4V(x + xxx)} = 3x^4 + 6xx - 1 = 0;$

Hinc $x^4 + 2xx = \frac{1}{3}$, $x^4 + 2xx + 1 = \frac{4}{3}$, & extrahendo radicem quadratam $xx + 1 = V\frac{4}{3}$ ſive $xx = V\frac{4}{3} - 1$, & rurſus extracta radice quadrata $x = V\overline{V\frac{4}{3} - 1}$.

PROBL. LXX.

Fig. 47. 207. Externæ portiones FD, fd ſecantium PD, Pd interceptæ peripheria AF quadrantis circularis, & tangente AD dividantur ad puncta H, b in qualibet ratione conſtanti, quæritur punctum flexus contrarii in curva AHb per hujuſmodi diviſionum puncta tranſeunte

Curva hæc eſt, ut patet, Conchois Nicomedea polo P, & regula, ſive aſymptoto BE dividente radium

Digitized by Google

dium A P in eadem ratione ad punctum B deſcripta. Quoniam enim $PA : AB = PD : DG = FD : DH$, etiam reliqua P F ad reliquam G H in eadem ratione erit ipſius $PA : AB$. Sed $PF = PA$, ergo $GH = AB$ conſtanti, ideoque AHb eſt Conchois polo P regula B E intervallo A B deſcripta. Jam vero ex puncto H ordinetur ad Conchoidem recta H K occurrens peripheriæ quadrantis circularis in puncto O, & jungatur O B, quæ erit parallela inclinatæ P H. Eſt enim $BO = BA = GH$, & H K parallela regulæ B E; Ergo erit $BK : KO = PK : KH$; Quare ſi BA ſit $= a$, $BP = b$, $BK = x$, $KH = y$, quoniam eſt $KO = V(aa - xx)$ erit $x : V(aa - xx) = c + x : y$, quare $y = \frac{(b + x) V(aa - xx)}{x}$. Eadem quoque erueretur æquatio, ſi ex puncto G duceretur recta G N parallela ipſi B A, eſſet enim recta $Nb = V(aa - xx)$ & cetera ut ſupra; Quare ſumtis differentiis, & mutatis ſignis, quibus afficitur dx, quippe creſcente y decreſcit x, erit $dy = \frac{xxxdx + aaadx}{xxV(aa - xx)}$. Fiat enim $V(aa - xx) = v$, erit $y = \frac{bv + vx}{x}$, & $xy = bv + vx$, quare $xdy - ydx = bdv - vdx + xdv$, & $dy = \frac{bdv - vdx + xdv + ydx}{x}$. Nunc loco v, & dv ſufficiantur eorum valores, erit $dy = \frac{bxdx}{xV[aa - xx]} - dxV(aa - xx) + \frac{xxdx}{xV(aa - xx)} + \frac{bdxV[aa - xx] + xdxV(aa - xx)}{xx} = \frac{xxxdx + aabdx}{xxV(aa - xx)}$, & ſumtis differentiis, poſitaque dx

conſtante,

conſtante, & ſuffecto v loco $\sqrt{(aa-xx)}$, cum ſit $dy = \frac{xxxdx + aabdx}{vxx} = \frac{xdx}{v} + \frac{aabdx}{vxx}$, erit $ddy = \frac{vdx^2 - xdxdv}{vv} - \frac{aabxxdxdv - 2aabvxdx^2}{vvx^4}$

$= \frac{vxxxdx^2 - x^4dxdv - aabxdxdv - 2aabvdx^2}{vvx^3}$,

ſuffectoque $\sqrt{[aa-xx]}$ loco v, & $-\frac{xdx}{\sqrt{(aa-xx)}}$ loco dv, erit $ddy = \frac{xxxdx\sqrt{(aa-xx)}}{vvx^4} +$

$\frac{x^5 dx^2 + aabxxdx^2}{vvx^4\sqrt{[aa-xx]}} - \frac{2aabdx^2\sqrt{(aa-xx)}}{vvx^4} =$

$\frac{-aaxxxdx^2 - 3aabxxdx^2 + 2a^4bdx^2}{(aa-xx)\sqrt{(aa-xx)}}$, mutantur enim ſigna ob x decreſcentem. Ponatur hæc quantitas $= 0$, erit $xxx + 3bxx - 2aab = 0$, quæ eſt æquatio ſolida tali ratione conſtruenda. Fiat BQ $\frac{1}{3}b + \frac{2aa}{3b}$, & vertice Q, axe vero QB, & parametro $= \frac{3}{2}b$ deſcribatur parabola Qb, quæ occurrat Conchoidi in puncto b, quod erit punctum flexus contrarii, ut facile demonſtrari poteſt, & ordinata bR $= yy = \frac{1}{2}bb + aa + \frac{3}{2}bx$.

CO-

COROLL.

208. Hinc ſi in æquatione $xxx + 3bxx = 2aab$ adjiciatur utrinque $bbb + 3bbx$, fiet $xxx + 3bbx + 3bbx + bbb = 2aab + bbb + 3bbx = 2byy$, nempe $PL^3 = 2 PB . Lb^2$. Quare ſi eadem manente diſtantia poli, nempe PB, variis intervallis BA, BV variæ conchoides ad eandem regulam BE deſcribantur, flexus eorundem contrarii *b*, S ſemper reperientur in curva parabolica cubica P*b*S ſecundi ordinis, cujus axis PC parallelus ipſi BE, parameter vero = 2PB, erit enim $PL^3 : PV^3 = Lb^2 : XS^2$. Vid. SLUSIUM.

PROBL. LXXI.

209. Externæ portiones FD, *fd* ſecantium PD, P*d* ex ſemicirculi termino P ad tangentem oppoſiti termini AD terminatarum dividantur in qualibet ratione conſtanti $n : m$ ad puncta H, *b*, quæritur punctum flexus contrarii in curva AH*b* per ſingula puncta diviſionum traducta. *Fig.* 47.

Hujuſmodi curva eſt, ut patet, Hyperciſſois, cujus aſymptotus BE diametrum AP in eadem illa ratione ſecat; ſemicirculi vero genitoris LKP diameter LP eſt portioni AB æqualis. Cum enim ſit PA : AB = PD : DG = FD : DH, & reliqua PF ad reliquam GH in eadem prorſus ratione erit ipſius PA : AB ſive LP, vel PF ad chordam PK, ergo GH ſemper = PK chordæ ſemicirculi LKP, adeoque AH*b* eſt Hyperciſſois, quam ſupra determinavimus. Jam vero dividatur AP in B in ratione $n : m$, deſcriptoque ſemicirculo AOB ordinetur recta HM eidem ſemicirculo occurrens in puncto O, ductaque recta OB fiat BGB parallela ipſi AD; Cum ſit PD : GD = PA : AB = $n : m$ = FD : DH, adeoque PF : GH = PA : AB, erit permutando PF : PA, = GH : AB. Sed

$PF : FA = PB : BG$, ob ſimilit. triang. AFP, BGP, ergo $BP : PG = GH : AB$, unde $PBA = PGH = Bg^2$. Sed $MB : HG = BP : PG = GH : BA$, quare $MBA = GH^2$, & $AB : BM = GH^2 : BM^2 = PH^2 : PM^2$; Hinc dividendo $AM : MB = HM^2 : PM^2$. Eſt autem $AM : MB = MO^2 : MB^2 = HM^2 : MP^2$; Quare OB ſubtenſa ſemicirculi eſt parallela ipſi PH. Sit jam $AP = a$, $AM = x$, $MH = y$, $AB = \frac{an}{m}$, nam $m : n = PA : AB$, erit $BM = \frac{an}{m} - x = \frac{an - mx}{m}$, $MP = a - x$, unde $\frac{an - mx}{m} : a - x = \sqrt{\frac{[anx - mxx]}{m}} : y$, & $y = \frac{[am - mx]}{an - mx} \sqrt{\frac{(anx - mxx)}{m}}$, quæ eſt æquatio curvæ propoſitæ. Quoniam vero æquationis hujuſce differentiæ nonniſi labore improbo ſumi poſſunt, neceſſum eſt eandem æquationem in aliam permutare, cujus differentiæ facilius eruantur, quod quidem obtinebitur, ſi loco $\frac{an}{m} = AB$ ponatur c, fiet enim $y = \frac{[a-x]\sqrt{[cx - xx]}}{c - x} = \frac{(a - x)\sqrt{x}}{\sqrt{[c - x]}}$. Ponatur $a - x = v$, & $\sqrt{[c - x]} = z$, erit $y = \frac{v\sqrt{x}}{z}$, & $yz = v\sqrt{x}$, ſumtiſque differentiis erit $ydz + zdy = \frac{vdx + 2xdv}{2\sqrt{x}}$, & $dy = \frac{vdx + 2xdv}{2z\sqrt{x}} - \frac{ydz}{z}$, factaque ſubſtitutione, erit $dy =$

$acdx$

$\frac{acdx - 3cxdx + 2xxdx}{2(c-x)\sqrt{(cx-xx)}}$. Fiat nunc $ac - 3cx + 2xx = v$, & $c - x = z$, erit $dy = \frac{vdx}{\sqrt{4xzzz}} = \frac{r}{t}$, ergo $ddy = \frac{tdr - rdt}{tt}$, hoc eſt, ſubſtituendo,

$$ddy = \frac{2zdxdv\sqrt{xz}}{4xzzz} - \frac{vdx}{4xzzz}\,\frac{(4zzzdx + 12xzzdz)}{2\sqrt{4xzzz}} =$$

$\frac{2xzdxdv - vzdx^2 - 3vxdxdz}{4xzz\sqrt{xz}}$, reductione facta ad eundem denominatorem; Fiant opportunæ ſubſtitutiones, erit $ddy = \frac{-3ccxdx^2 - accdx^2 + 4acxdx^2}{4x[c-x]^2\sqrt{[cx-xx]}}$ $= 0$, unde $-3ccx - acc + 4acx = 0$, & $4acx - 3ccx = acc$, ſive demum $x = \frac{ac}{4a - 3c}$.

PROBL. LXXII.

210. Invenire formulam, cujus ope determinari poſſit punctum flexus contrarii in curvis, quarum ordinatæ a polo, ſive fixo puncto procedunt. Fig. 49.

Sit curva ABC, cujus polus P, a quo prodeant ordinatæ PB, P*b*, quæ ſint inter ſe infinite proximæ. Ducta PT normali ipſi PB, & P*t* normali ipſi P*b* agantur tangentes BT, B*t*, quarum altera occurret normali PT in puncto T, altera vero normali P*t* in puncto *t*, & centro P intervallis PT, PB deſcribantur arculi TF, BD, qui quidem erunt infinite parvi, ſi quidem infinite parvi ſunt ang. TPF,

BPD. His poſitis patet in perimetro concava rectam
Num. 1. Et diverſa poſitione locatam eſſe relate ad punctum
P, ac locetur in perimetro convexa, ibique eſſe Pt
Num. 2. $> PT$, & contra. Igitur in tranſitu curvæ a concavitate ad convexitatem, & contra; nempe in puncto flexus contrarii quantitas Et ex poſitiva fiet negativa; & e contrario, adeoque tranſibit vel per o, vel per oo, ac proinde æquanda erit nihilo, vel infinito.

Jam vero per conſtructionem ſimilia ſunt triang. bBD, BPT, ut patet; item ſimilia ſunt triang. BPT, TFE; nam TF, & PB ſunt inter ſe parallelæ, quare ang. $FTE = PBT$, & ang. ad F, & P ſunt recti. Sunt quoque ſimiles inter ſe ſectores PBD, PTF, nam ang. ad D, & F ſunt recti, & ang. $TPF = BPD$, ſi enimvero ab æqualibus tPb, TPB auferatur communis BPT, reliqui erunt æquales. Sit nunc $PB = y$, $BD = x$; Et quoniam $bD : DB = bP$, vel $BP : PT$, erit $dy : dx = y : \frac{ydx}{dy} = PT$. Præterea quoniam eſt $PB : BD = PT : TF$, erit $y : dx =$
Num. 1. $\frac{ydx}{dy} : \frac{dx^2}{dy} = TF$. Denique cum ſit bP, vel BP : PF, vel $PT = TF : FE$, erit $y : \frac{ydx}{dy} = \frac{dx^2}{dy} : \frac{dx^3}{dy^2} = FE$. Sed tF eſt differentia ipſius PT, ergo ſi dx ſumatur conſtans, erit $FE = \frac{dxdy^2 - ydxddy}{dy^2}$, proindeque $tE = tF + FE = \frac{dxdy^2 - ydxddy + dx^3}{dy^2}$, qui valor æquari debet vel o, vel oo; Quare multipli-

catione

catione inſtituta per dy^2, & diviſione per dx, erit $dy^2 + dx^2 - yddy = o$, vel oo, quæ eſt formula ad punctum flexus contrarii determinandum in curvis ad polum, vel fixum punctum relatis.

Idem dicendum de curva convexa; cum enim in ea quantitas tE ſit negativa, erit eadem $= -\dfrac{dxdy^2 + ydxddy - dx^3}{dy^2}$, adeoque $yddy - dx^2 - dy^2 = o$, vel oo. Num. 2.

COROLL.

211. Hinc patet, quod ſi y ponatur infinita evaneſcent in formula $dy^2 + dx^2 - yddy$ termini $dy^2 + dx^2$ relate ad $yddy$; Quare $-yddy = o$, vel oo, & dividendo per $-y$, erit $ddy = o$, vel oo, quæ eſt formula pro flexu contrario determinando in curvis, quarum ordinatæ ſunt inter ſe parallelæ. Et quidem poſita y infinita, ordinatæ ſunt, vel haberi poſſunt ut inter ſe parallelæ.

PROBL. LXXIII.

Invenire punctum flexus contrarii in Conchoide Nicomedea, in qua ordinatæ Pt, PT ex polo, ſive fixo puncto P procedunt.

Ducatur PT infinite proxima ipſi Pt, & centro P intervallis Pt, Pb deſcribantur arculi infinite parvi tr, bO. Sit $AB = a$, $PB = b$, $Pt = y$, $Pb = z$, $tr = dx$, erit ergo $bt = y - z$, adeoque cum bt ob naturam Conchoidis ſit $= AB$, invenietur ejuſdem æquatio $y - z = a$, proindeque ſumtis differentiis $dy = dz$. Cum autem ſimilia ſint triang. bBP, HbO, Fig. 9.

Digitized by Google

erit $bB : BP = HO : Ob$, $\sqrt{(zz-bb)} : b = dz : \frac{bdz}{\sqrt{(zz-bb)}} = Ob$. Item quoniam ſimilia ſunt triang. PbO, Ptr, erit $Pb : Pt = bO : tr$, $z : z+a = \frac{bdz}{\sqrt{(zz-bb)}} : \frac{bzdz+abdz}{z\sqrt{(zz-bb)}} = tr$. Sed $tr = dx$, ergo $dx = \frac{bzdz+abdz}{z\sqrt{[zz-bb]}}$, adeoque $\frac{zdx\sqrt{(zz-bb)}}{bz+ab} = dz = dy$, cujus æquationis differentia, poſita dx conſtante, erit $ddy = \frac{(bzzz+2abzz-abbb)dxdz}{(bz+ab)^2\sqrt{(zz-bb)}}$.

Sed $dz = \frac{zdx\sqrt{(zz-bb)}}{bz+ab}$, ergo loco dz hoc ipſius valore ſuffecto, erit $ddy = \frac{[bz^4+2abzzz-abbbz)dx^2}{[bz+ab]^3}$.

Quare ſi in formula generali dy^2+dx^2-yddy loco dy^2, y, & ddy ſufficiantur eorundem valores ſupra inventi, fiet eadem $= \frac{[z^4-bbzz]dx^2}{[bz+ab]^2} + dx^2 - \frac{[z^4-2azzz+abbz]dx^2}{[bz+ab]^2} = \frac{(3abbz+aabb-2azzz)dx^2}{[bz+ab]^2} = 0$, atque hinc

$$zzz - \frac{3}{2}bbz - \frac{1}{2}abb = 0.$$

Si fuerit

Si fuerit $a = b$, fiet æquatio $zzz - \frac{3}{2}aaz - \frac{1}{2}aaa = 0$, & diviſione inſtituta per $z + a$, fiet $zz - az - \frac{1}{2}aa = 0$, ſive $zz - az + \frac{1}{4}aa = \frac{3}{4}aa$, & $z = \frac{1}{2}a + \sqrt{\frac{3}{4}aa}$.

PROBL. LXXIV.

212. Invenire punctum flexus contrarii in curva ejus naturæ; ut ductis a polo, ſive fixo puncto P ordinatis Pt, PT &c. ſit ſemper rectang. $Pbt = PBA$.

Ponatur jam factum quod quæritur, & ductis inter ſe infinite proximis Pt, PT centro P intervallis Pb, Pt deſcribantur arculi infinite parvi bo, tr. Sit $AB = a$, $PB = b$, $Pt = y$, $Pb = z$, $tr = dx$, erit ergo $bt = y - z$, & curvæ æquatio fiet $yz - zz = ab$, proindeque $y = \frac{ab + zz}{z}$, & $dy = \frac{zzdz - abdz}{zz}$.

Eſt autem ob ſimilitudinem triang. bBP, Hbo, $bB : bP = Ho : ob$, $\sqrt{(zz - bb)} : b = dz : \frac{bdz}{\sqrt{(zz - bb)}} = ob$, & ob ſimilit. triang. bPo, tPr, $bP : Pt = ob : tr$, $z : y = \frac{bdz}{\sqrt{[zz - bb]}} : \frac{bydz}{z\sqrt{[zz - bb]}} = tr$, ergo $dx = \frac{bydz}{z\sqrt{(zz - bb)}}$, & loco y poſito ejus valore, erit $tr = \frac{abbdz + bzzdz}{zz\sqrt{[zz - bb]}} = dx$; hinc $dz = \frac{zzdx\sqrt{(zz - bb)}}{abb + bzz}$.

Ponatur hic valor in ſupra invento valore ipſius dy, erit $dy = \frac{zzdx\sqrt{[zz - bb]} - abdx\sqrt{[zz - bb]}}{abb + bzz}$.

Sumatur

Sumatur æquationis hujuſce differentia, ponendo dx conſtantem, erit $ddy = \frac{(4abbzzz + bz^5 - 4ab^4 z - aabbbr)\,dx\,dz}{(abb + bzz)^2 V(zz - bb)}$. Reſtituatur in locum dz ejus valor, erit $ddy = \frac{[4abbz^5 + bz^7 - 4ab^4 z^3 - aab^3 z^3]\,dx^2}{[abb + bzz]^3} = 0$, proindeque ſi in formula $dy^2 + dx^2 - yddy = 0$ fiant neceſſariæ ſubſtitutiones, & ordinetur æquatio, fiet $z^4 + \frac{2}{3}abzz - \frac{4}{3}bbzz - \frac{4}{3}abbb - \frac{1}{3}aabb = 0$.

PROBL. LXXV.

Fig. 50. 213. Sit portio circuli $ABbC$, cujus radius PA, ſitque curva $ADaF$ ejus naturæ, ut ducto a quovis puncto B arcus circuli radio BP ſecante curvam in D ſit ſemper arcus $AB : BD = BD : a$, quæritur punctum flexus contrarii D.

Ex puncto P ducatur radius Pb infinite proximus radio PB, & centro P intervallo PD deſcribatur arculus DE. Sit autem PA, vel $PB = r$, ejus portio $PD = y$, erit ergo $BD = r - y$; Sit præterea $AB = z$, & $DE = dx$, eritque ob naturam curvæ $az = rr - 2ry + yy$, quare, ſumtis differentiis, erit $adz = -2rdy + 2ydy$, ſive $dz = \frac{2ydy - 2rdy}{a} = Bb$.

Jam vero ob ſimilitud. ſectorum PBb, PDE, erit $PB : PD = Bb : DE$, $r : y = \frac{2ydy - 2rdy}{a} : dx = \frac{2yydy - 2rydy}{ar}$, cujus differentia, poſita dx conſtante,

ſtante, erit $4ydy^2 + 2y^2 ddy - 2rdy^2 - 2ryddy$; hinc $yddy = \frac{rdy^2 - 2ydy^2}{y-r}$. Igitur ſi in formula generali $dy^2 + dx^2 - yddy$ loco dx^2, & $-yddy$ ſuſſiciantur eorum valores, erit $dy^2 + dx^2 - yddy = dy^2 - \frac{4y^4 dy^2 - 8ry^3 dy^2 + 4rryydy^2}{aarr} - \frac{rdy^2 + 2ydy^2}{y-r} = 0$, hoc eſt neceſſariis inſtitutis operationibus $4y^5 - 12ry^4 + 12rry^3 - 4rrryy + 2aarry - 2aarrr = 0$, qua reſoluta æquatione obtinebitur valor ipſius P D.

CAP. V.

De uſu Calculi Differentiarum ſecundarum in determinandis evolutis, & ſubevolutis curvarum, & radio oſculatore.

DEF.

214. SI curva A B C D in ſua perimetro convexa filo aliquo, quod ſit perfecte flexibile, & diſtractionis incapax, convolvatur, filum autem ſit fixum in D, liberum vero in aliis curvæ punctis, & ab extremitate curvæ ipſius A filum exſolvatur uſque in D, ita ut perimeter curvæ A B C D filo ipſo explicetur, patet extremitate A fili ſe exſolventis & in rectas B E, C F, D G ſe ſe componentis generari curvam aliam in G terminatam, nimirum curvam A E F G, *Fig. 51.*

quam

quam deinceps curvam *ex evolutione genitam*, vel *lineam evolutionis* appellabimus. Curva autem A B C D dicetur *evoluta* curvæ A E F G, & punctum A ejuſdem *origo*, punctum vero G *terminus*, ſi punctum D terminus fuerit ipſiuſmet evolutionis. Demum quælibet ex rectis B E, C F &c. dicetur *radius evolutæ*, vel *radius curvedinis*, ſive etiam *radius oſculator*, quo nomine veniunt etiam *circuli* radiorum oſculantium B E, C F &c.

COROLL. I.

215. Conſtat autem ex geneſi curvæ A E F G, quod ſi arcus A B curvæ A B C D defuiſſet, *nimirum* ſola curva B C D filo fuiſſet convoluta, ad cujus extremum B fili A B C D portio A B tangens fuiſſet, in puncto nempe, ubi incipit evolvi, extremitas hujuſce fili ſe exſolventis uſque in D eundem curvæ arcum E G, ut ſupra, deſcripſiſſet, cui quidem curvæ, ut & curvæ B C D eædem, quæ ſupra, definitiones conveniunt. *Fig. 52.*

COROLL. II.

216. Illud etiam conſequitur ex geneſi curvæ A E F G, quod ubi extremitas A dicti fili erit in quolibet puncto E curvæ A E F G, portio ipſius B E ſemper erit æqualis arcui ipſius evoluto A B. Eſt enim ob adplicationem fili E B C D ſupra curvam integram, vel ejuſdem arcum A B C D fili ipſius longitudo æqualis longitudini hujuſce arcus A B C D, eademque ratione cum portio altera B C D ejuſdem fili, quo convolvitur portio B C D curvæ A B C D ſit æqualis huic portioni B C D dictæ curvæ, recta F C æquari debet arcui evoluto A B C, & recta G D toti curvæ A B C D; Quare ſemper habebitur $EB + BC = ABC = FC$, & $EB + BCD = ABCD = GD$.

COROLL.

COROLL. III.

217. Patet præterea, quod ſi ſola curva B C D filo fuiſſet convoluta, portio curvæ B C æqualis eſſet differentiæ rectarum B A, C E, & portio C D æqualis differentiæ rectarum D F, C E, tota vero curva B C D æqualis differentiæ rectarum F D, A B.

COROLL. IV.

218. Quilibet radius eſt ſemper tangens evolutæ. Cum enim radius ſit productio alterius ex minimis lateribus, ex quibus curva componi intelligitur, radius ipſe tanget curvam, nec eam unquam ſecabit.

COROLL. V.

219. Ulterius ſumto arcu E F infiniteſimo primi ordinis cujusvis curvæ, ductiſque normalibus E H, F H ad curvam, ſi centro H intervallo F H deſcribatur arculus F K, cadet hic intra curvam A E F verſus E. Eſt enim ob curvæ A E F G geneſim recta E B æqualis curvæ A B, & recta F C æqualis curvæ F C. Sunt autem tangentes infiniteſimæ B H, C H ſimul majores elemento curvæ B C, quare rectæ E H, H C ſimul ſumtæ majores erunt curva A B C, nempe recta F C, & ablata communi H C erit E H major recta F H, unde circulus F K centro H, & radio H F deſcriptus cadet intra curvam.

COROLL. VI.

220. Patet denique ex dictis nullam dari poſſe evolutionem integram curvæ, quæ puncto flexus contrarii donetur. Si enimvero id fieri poſſet, neceſſum eſſet, ut in puncto flexus contrarii filum ſe ſe exſolvens evolutam transgrederetur, nempe motu moveretur illi contrario, quo ante punctum ipſum movebatur. Cum

Digitized by Google

autem hujuſmodi motus dici non poſſit continuus, conſtat apertiſſime duas curvæ portiones, quas extremitas fili ſe exſolventis deſcribit, tum antequam ad punctum flexus contrarii pertingat, tum poſtquam punctum ipſum fuerit prætergreſſa non modo ad unam eandemque lineam pertinere non poſſe, ſed ne ad unum quidem, idemque crus curvæ.

COROLL. VII.

221. Unde conſtat exiſtere nullo modo poſſe punctum illud, quod Cl. HOSPITALIUS, ceterique omnes de re Algebrica Scriptores vocant punctum *regreſſus ſecundæ ſpeciei*, adeoque ineptas, ac prorſus ſupervacaneas eſſe formulas illas, quas iidem Auctores ad ejus puncti inventionem ſtabilierunt.

THEOREMA.

Fig. 53. 222. Si ſit quævis curvæ portio A B infiniteſima primi ordinis, & a punctis A, B ducantur rectæ A D, B D normales ad curvam, hujuſmodi rectæ pro æqualibus haberi poterunt.

Secus enim ponatur D B > A D, & ex puncto A ducatur A E normalis ad B D; Et quoniam ang. ad A eſt rectus, & ab ang. recto cadit in baſim normalis A E, erit D E : E A = E A : E B, nempe finita quantitas ad infiniteſimam primi ordinis, ut hæc ad infiniteſimam ſecundi, proindeque ipſæ A D, B D quippe quæ a ſe invicem diſcrepant quantitate, quæ conſiderari poteſt ut zero, pro æqualibus habendæ erunt, & triang. A B D pro iſoſcele reputandum.

COROLL. I.

223. Unde profluit, quod poſito arcu circuli infiniteſimo primi ordinis A B, differentia inter ſinum rectum, & tangentem erit infiniteſima tertii. Sunt enim triang.

triang. DEA, AFG ſimilia inter ſe; quare erit DE : EA = AF : FG, nempe finita quantitas ad infiniteſimam primi ordinis, ut infiniteſima ſecundi ad infiniteſimam tertii.

COROLL. II.

224. Hinc colligitur, quod cum tangens ſit ſemper major arcu, arcus chorda, & chorda ſinu recto, quoniam adſumi poſſunt pro æqualibus tangens, & ſinus rectus, cum non differant inter ſe niſi quantitate infiniteſima tertii ordinis, pro æqualibus adſumi poterunt tangens, arcus, chorda, & ſinus rectus.

COROLL. III.

225. Quoniam vero oſtenſum eſt triang. ABD eſſe iſoſcelion, patet arculum quemvis curvæ cujuslibet infiniteſimum iiſdem omnino proprietatibus donari cum arcu circuli infiniteſimo centro D, & radio DA, vel DB deſcripto; Quare tangentes AC, CB erunt inter ſe æquales, & recta DC normalis erit rectæ AB, eamque biſariam ſecabit in puncto I.

COROLL. IV.

226. Præterea quoniam ob ſimilit. triang. DIA, AIC eſt $DI : IA = IA : IC = \frac{IA^2}{DI}$, & ob circuli naturam eſt $IA^2 = 2DI \cdot IO = DI \cdot IC$, erit $2DI : DI = IC : IO$, adeoque IO, OC inter ſe æquales.

COROLL. V.

227. Hinc ſi centro A intervallo AI deſcribatur arculus circuli IH, erunt triang. AIH, ICH inter

ſe ſimilia ob ang. rectos ad puncta I, H, & communem in puncto C, quare AC : CI = CI : CH, & eſt AC infiniteſima primi ordinis, CI infiniteſima ſecundi; Quare erit CH infiniteſima tertii, adeoque dimidium chordæ AI, ſive etiam arculus curvæ AO differet a tangente AC unica tantum infiniteſima tertii ordinis.

THEOREMA.

Fig. 51. 228. Sumto arcu EF cujuslibet curvæ infiniteſimo primi ordinis, ductiſque normalibus EH, FH ad curvam, & centro H intervallo FH deſcripto arculo FK, qui quidem cadet intra curvam AEF verſus E, intercepta EK erit quantitas infiniteſima tertii ordinis.

Cum enim tangentes BH, HC ſint majores arculo BC quantitate tertii ordinis infiniteſima, manifeſtum eſt curvam AB + BH + HC majorem eſſe curva ABC ſive recta FC eadem ipſa infiniteſima, quare ablata communi HC, erit AB + BH, nempe EH major ipſa FH quantitate infiniteſima tertii ordinis.

COROLL. I.

229. Unde conſtat circuli arculum FK confundi cum arculo curvæ FE, adeoque ipſorum alterum pro altero adſumi poſſe, & tangentem FC normalem eſſe curvæ AEF in puncto F, & GB eidem normalem in puncto E.

COROLL. II.

230. Quoniam vero poſito arcu curvæ EF infiniteſimo ipſæ EH, FH ſumi poſſunt pro æqualibus, cum non differant inter ſe niſi infiniteſima tertii ordinis, & ipſæ EB, BH a ſe invicem diſcrepent ipſa BH infiniteſima primi ordinis, quandoquidem & arcus BC evolutæ eſt infiniteſimus ejuſdem primi ordinis, & eſt EB quantitas finita, ergo ipſæ EB, BH pro æqualibus adſumi poterunt.

CO-

COROLL. III.

231. Hinc patet, quod ut determinetur quodvis punctum B evolutæ, ſive longitudo cujuſvis radii oſculatoris E B, data normali E H poſitione inveniri debet punctum H, in quo eadem E H ſecatur cum normali infinite proxima F H, quod qua ratione præſtari poſſit infra declarabitur.

THEOREMA.

232. Curvæ per fili explicationem genitæ curvatura maxima eſt, ubi incipit evolutio, minima vero, ubi deſinit.

Circulorum enim curvitates ſunt reciproce ut radii, a quibus circuli ipſi deſcribuntur: Circulus quippe radii longioris minorem habet curvaturam, quam circulus radii brevioris quippe in ampliori circulo diſpergitur, quæ in anguſtiori magis colligitur curvaturæ ſpecies. Atqui quilibet arcus curvæ genitæ per fili explicationem nihil eſt aliud, ſi minimus accipiatur, quam arcus circuli, cujus radius eſt ipſemet evolutæ radius, & radius evolutæ minimus eſt, ubi incipit evolutio, maximus ubi deſinit, nimirum curva per fili explicationem genita conſtat ex infinitis hiſce arcubus circulorum minimis, quorum radii numero infiniti radii evolutæ, ergo patet propoſitum.

THEOREMA.

233. Quælibet curva, & cujuſvis ea ſit naturæ haberi poteſt pro linea evolutionis alterius curvæ, quæ ejuſdem eſt evoluta.

Quoniam enim ad omnia cujuſvis curvæ puncta duci poſſunt tangentes, ita ad ſingula quæque cujuſvis curvæ puncta erigi poſſunt normales, quæ alicubi ſe ſecabunt, cum non ſint inter ſe parallelæ, quare ſi inter ſe infinite proximæ concipiantur, ſingula quæque

puncta

puncta concurſus duarum normalium infinite proximarum erunt puncta alterius curvæ, quæ erit evoluta.

THEOREMA.

234. Si linea evolutionis fuerit curva geometrica cognita, longitudines radiorum evolutæ, ejuſque quadratura, & proprietates determinari poterunt.

Quod patet; Cognito enim radio cognoſcitur & arcus idem reſpondens.

THEOREMA.

235. Diſtantia centri oſculatoris circuli ab ordinata curvæ eſt ad unitatem ut tertia proportionalis elementorum abſciſſæ, & curvæ eſt ad elementum rationis elementorum abſciſſæ, & ordinatæ. *Fig. 54.*

Accipiatur in perpendiculari CG quodvis punctum G, ex quo ducatur GF normalis ad axem. Sitque $AF = a$, $GF = b$, $AB = x$, $BC = FH = y$, erit $CO = dx$, $Oc = dy$, Et quoniam triang. GHC, & COc ſunt ſimilia erit $GH : HC = CO : Oc$, $b + y : a - x = dx : dy$, & $b + y = \frac{(a-x)dx}{dy}$; Ubi autem punctum G eſt ad evolutam, eadem manet tum a, tum b, cum utraque CG, cG in idem commune punctum conveniant, ideoque ut conſtans tractari debet; itaque ſumtis differentiis poſtremæ æquationis poſitaque dy conſtanti, erit $dy = \frac{addx - xddx - dx^2}{dy}$,

ſive $dy^2 = (a-x)ddx - dx^2$, vel $dy^2 + dx^2 = (a-x)ddx$; Sive $ds^2 = [a-x]ddx$, & $\frac{ds^2}{dy} = \frac{[a-x]ddx}{dy}$, hoc eſt $\frac{ds^2}{dy} : \frac{ddx}{dy} = a - x : 1$, nempe

tertia

Digitized by Google

tertia proportionalis poſt dy, & ds eſt ad elementum rationis $\frac{dx}{dy}$, quod eſt $\frac{ddx}{dy}$, ut diſtantia centri circuli oſculatoris ab ordinata ad unitatem, ut invenit LEIBNITIUS *.

* *Act. Lypſ.* 1694.

THEOREMA.

236. Radius oſculi eſt ad unitatem, ut elementum unius coordinatæ eſt ad elementum rationis alterius coordinatæ, & curvæ.

Ponatur radius GC $= r$, eritque $r : a - x = ds : dy$, ſive $\frac{rdy}{ds} = a - x$, & ſumtis differentiis, poſitaque ds conſtanti, $\frac{rddy}{ds} = -dx$, itaque $r : 1 = -dx : \frac{ddy}{ds}$, proindeque $\frac{dsdx}{ddy} = r$, ut JACOB. BERNOULLIUS invenit.

PROBL. LXXVI.

237. Data linea evolutionis A C c, cujus ſemiordinatæ MC ſint axi AR perpendiculares, & data qualibet ad curvam normali PC, determinare valorem radii PC evolutæ ejuſdemmet curvæ A C c, ſive concurſum normalium PC, P c inter ſe infinite proximarum. *Fig.* 55.

Ex puncto c ducatur recta cm normalis axi AR, quæ erit infinite proxima rectæ CM ſiquidem punctum c ponitur infinite proximum puncto C. Ex puncto P concurſus ducatur recta PG axi AR parallela, occurrens in punctis G, g ordinatis CM, cm productis. Ponatur CG $= z$, AM $= x$, MC $= y$, erit ergo Gg = Mm = CH $= dx$, cH $= dy = dz$. Quoniam vero ang. GCH = PCc, quippe ſunt ambo recti, ſi auferatur communis PCH, erit ang. GCP = HCc, & ang. ad G, & H ſunt recti ac proinde æquales, quare triang. CGP, CHc ſunt inter ſe ſimimilia,

milia, proindeque CH : Hc = CG : GP, $dx : dy = z : \frac{z\,dy}{dx}$, adeoque FP = AM + GP = $x + \frac{z\,dy}{dx}$. At hic valor, ut & valor ipſius AF perſeverat idem, ſi ſpectetur radius CP, vel cP, ergo ejuſdem differentia æqualis nihilo ſit oportet, erit igitur

$$\frac{dx^3 + z\,dx\,ddy + dx\,dy\,dz - z\,dy\,ddx}{dx^2} = o,$$

hinc $z = \frac{dx^3 + dx\,dy\,dz}{dy\,ddy - dx\,ddy}$. Eſt autem CH : C$c$ = CG : CP, $dx : dt = z : \frac{z\,dt}{dx}$, ergo $CP = \frac{dx^2\,dt + dy\,dz\,dt}{dy\,ddx - dx\,ddy}$.

Atqui $dy = dz$, quare $CP = \frac{dx^2\,dt + dy^2\,dt}{dy\,ddx - dx\,ddy} = \frac{dt^3}{dy\,ddx - dx\,ddy}$.

Si ergo dx ponatur conſtans, vel $ddx = o$, fiet $CP = \frac{dt^3}{-dx\,ddy}$.

Si ponatur conſtans dt, CP erit $= \frac{dt\,dy}{ddx}$; Quoniam enim $dx^2 + dy^2 = dt^2$, ſumtis differentiis erit $2dx\,ddx + 2dy\,ddy = o$; Quare $ddy = -\frac{dx\,ddx}{dy}$, proindeque $CP = \frac{dt^3}{dy\,ddx - \frac{dx^2\,ddx}{dy}} = \frac{dt\,dy}{ddx}$, poſito dt^2 loco $dy^2 + dx^2$.

Si dy ponatur conſtans, erit $CP = \frac{dt^3}{dy\,ddx}$.

Digitized by Google

Quod ſi loco dt^2 ponatur $dx^2 + dy^2$, erit

$$\frac{dt^3}{-dxddy} = \frac{(dx^2+dy^2)^{\frac{3}{2}}}{-dxddy},$$ quæ eſt formula HOSPITALIANA radii oſculatoris, & ſic de ceteris formulis.

COROLL.

238. Hinc eruitur, quod ſi fuerit curva quælibet, cujus coordinatæ x, y, ejuſque differentialis æquatio $dx = vdy$ [per v intelligatur quantitas ex x, y, & conſtantibus utcumque coaleſcens,] poſito dt elemento curvæ, erit radius oſculi $\frac{dt^3}{dy^2\,dv}$; Sumta enim dy conſtante, cum ſit $dx = vdy$, erit $ddx = dvdy$, adeoque ſuffecto loco ddx ipſius valore $dvdy$, prodibit radius $\frac{dt^3}{dy^2\,dv}$. Sic ubi $ax = yy$ adeoque $adx = 2ydy$, fit $dx = \frac{2ydy}{a}$, & $v = \frac{2y}{a}$, unde $dv = \frac{2dy}{a}$, itaque radius oſculi $\frac{adt^3}{2dy^3}$. Sic autem conſtruitur formula $\frac{dt^3}{dy^2\,dv}$. Ex puncto T, ubi tangens curvæ AC occurrit axi, erecta normali TG, quæ ſecet normalem ad curvam in puncto G ſi conſtruatur nova curva ZP ex quartis proportionalibus BP poſt BC, BT, & conſtantem V ad arbitrium adſumptam, & ducatur tangens PM novæ curvæ, mox fiat TB : *Fig. 56.*

$BM = GE : CD$, erit CD radius oſculi. Eſt enim $\frac{BM \cdot GE}{TB} = CD$, & ob ſimilit. triang. GTE, TBC eſt $CB : CT = TE : GE = \frac{CT \cdot TE}{CB}$. Sufficiatur hic valor ipſius GE in illius locum, erit $\frac{BM \cdot CT \cdot TE}{TB \cdot CB} = CD$. Sunt autem ſimilia inter ſe triang. TBC, TCE, quare erit $TB : CT = CT : TE = \frac{CT^2}{TB}$; quare erit $CD = \frac{BM \cdot CT^3}{TB^2 \cdot CB}$. Atqui ex conſtructione eſt $V : PB = BC : BT$, adeoque $\frac{PB \cdot BC}{V} = BT$; igitur præcedente valore in hujus locum ſuffecto, erit $CD = \frac{BM \cdot CT^3 \cdot V}{PB \cdot CB^2 \cdot BT} = \frac{dx \cdot dt^3 \cdot V}{dx \cdot dy^2 \cdot V \cdot dv} = \frac{dt^3}{dy^2\, dv}$.

Huic fere ſimilem conſtructionem præbet Analyſta plane ſummus JOH. BERNOULLIUS. Quinimo eadem conſtructio obtineri poteſt independenter a formula radii oſculatoris $\frac{dt^3}{dy^2\, dv}$. Sit enim vero AC curva propoſita, cujus tangens CT in puncto C, & ct in puncto c alteri C infinite proximo. Sumatur extra curvæ axem quodlibet punctum Q, per quod ducatur recta QK parallela tangenti CT, & Qk parallela itidem alteri tangenti ct, & facto centro Q intervallo QK deſcribatur arcus KL, & ad abſciſſam AB propoſitæ curvæ deſcribatur alia curva ZS ea lege,

ge, ut producta ordinata CB ſit $BP = AK$, & $bp = Qk$ &c. curvæ autem ZS tangens in puncto P ſit recta PM, erit $BT : BM = GE : CD$ radium oſculi in puncto C. Eſt quidem $LK : Kk = QA : QK = CB : CT$; pR (Kk) : $PR = BP : PM = AK : BM$; PR (CO) : $Cc = BT : CT = CT : TE$, quare ſi ducantur in ſe mutuo hujuſmodi proportionum termini homologi, erit $KL : Cc = CB . AK : BM . TE$. Et quoniam QK, Qk ſunt parallelæ tangentibus TC, tc, erit ang. $KQk = Tct$. Sed huic æquatur etiam ang. C Dc, quare ſimiles erunt ſectores QKL, C Dc, adeoque erit $KL : Cc = QK : CD = CB . AK : BM . ET$, proindeque $QK . BM . ET = CD . CB . AK$, & ſubſtituendo ipſarum QK, KA proportionales CE, & CB, fiet $CE . BM . ET = CD . CB^2$, & rurſus ſubſtituendo GE, TE loco proportionalium CE, BE, fiet $GE . BM . TE = CD . TB . TE$, hoc eſt $BM . CE = CD . TB$, unde habetur $BT : BM = GE : CD$.

Si ordinatæ BP ipſis QK æquales fierent, & ex termino M ſubtangentis erigeretur recta MH normalis ad axem occurrens normali EC productæ, radius oſculi CD æqualis eſſet portioni CH extra datam curvam. In hoc ſiquidem caſu fieret $LK : Cc$, vel $QK : CD = QA : BM$, & permutando $QK : QA = CD : BM$; ſed $QK : QA = CE : BE = HC : MB$, quare $HC : BM = CD : BM$, proindeque $CD = HC$.

Quod ſi æquatio differentialis curvæ ſit $dy = v\,dx$ adeoque poſita dx conſtante, ſit $ddy = dv\,dx$, erit radius oſculi $= \dfrac{dt^3}{-dx^2\,dv}$.

Si ſumatur conſtans dy, erit radius oſculi $= \dfrac{dt^3}{dy\,ddx} = \dfrac{dt^3}{dy^2\,dv}$.

 Denique,

Denique, ſi ponatur conſtans dt, erit radius $= \frac{dx\,dt}{-ddy}$, vel $= \frac{dy\,dt}{ddx} = \frac{dt}{dv}$.

COROLL.

239. Quoniam vero radius ad ſuboſculatricem eſt ut elementum curvæ ad elementum abſciſſæ, nempe ut $dt : dx$, erit ſuboſculatrix $= \frac{dt^2}{-ddy}$. *Suboſculatrix* autem, vel *Coradius* dicitur recta CG, quæ ab extremitate radii oſculatoris C normalis ducitur ad axem AR occurrens in puncto G rectæ PG, quæ ab altero extremo ejuſdem radii parallela ducitur eidem axi AR.

PROBL. LXXVII.

240. Parabolæ ACc ſuboſculatricem CG, vel radium oſculatorem in quovis puncto C invenire.

Sit abſciſſa $AM = x$, $MC = y$, parabolæ parameter $= a$, eritque ejuſdem æquatio $yy = ax$, quare $a\,dx = 2y\,dy$, & ſumta dx conſtante, $o = 2dy^2 + 2y\,ddy$, adeoque $ddy = -\frac{dy^2}{y}$. Eſt autem $dy = \frac{a\,dx}{2y}$, ergo $dy^2 = \frac{aa\,dx^2}{4yy}$, & per conſequens $ddy = -\frac{aa\,dx^2}{4yyy}$. Sufficiantur hi valores in formula ſuboſculatricis, fiet

$$\frac{dt^2}{-ddy} = \frac{4yyy}{aa} + y = \frac{4x\sqrt{ax}}{a} + \sqrt{ax} = CG.$$

Ex puncto C ducatur tangens CT occurrens axi in puncto T, a quo ſi ducatur TG parallela normali
Fig. 55. CK, occurret ipſa rectæ MC productæ in puncto quæſito G. Cum enim ang. CTG ſit rectus, erit CM :

MT

MT = MT : MG, hoc eſt, ob parabolam, $\sqrt{ax} : 2x = 2x : \frac{4xx}{\sqrt{ax}} = \frac{4yyy}{a} = \frac{4ax\sqrt{ax}}{aa} = \frac{4x\sqrt{ax}}{a}$; Quare CM + MG, hoc eſt $CG = \frac{4x\sqrt{ax}}{a} + \sqrt{ax}$.

Quod ſi determinata CG ducatur GP axi AR parallela occurret hæc normali CP in puncto P, quod erit ad evolutam.

Aliter. Eſt ob ſimilit. triang. CMK, CGP, $CM : MK = CG : GP$, $\sqrt{ax} : \frac{1}{2}a = \frac{4x\sqrt{ax}}{a} + \sqrt{ax} : 2x + \frac{1}{2}a = GP = RM$, ergo $KR = 2x$, adeoque ſumta KR dupla ipſius AM, vel MR = TK, ductaque RP parallela ipſi CM occurret hæc normali CK productæ in puncto P; quod erit ad evolutam. Quoniam ergo eſt CM : CK = CG : PC, erit $\sqrt{ax} : \frac{\sqrt{[4ax+aa]}}{2} = \frac{4x\sqrt{ax}}{a} + \sqrt{ax} : \frac{[4ax+aa]\sqrt{[4ax+aa]}}{2aa} = \frac{[4x+a]\sqrt{[4x+a]}}{2\sqrt{a}} = CP$ radio oſculatori.

Juxta priorem methodum. Sit AC parabola; erit $BC : CT = BE$ [$= \frac{1}{2}$ parametri = V conſtanti] : *Fig.* 57.
BC, quare ipſa BC = BP, ejuſque ſubtangens BM = BT; cum autem ſit BT = BM, erit EG = CD, & ablata communi CE, ſit reliqua GC = ED, quod convenit cum ſuperiore conſtructione, quæ eſt Clariſs. HOSPITALII *pag.* 81. Siquidem ex occurſu tangentis ducta T*H* normali parallela, atque hinc *HD* parallela ad axem, habetur *HT* = ED = CG.

PROBL. LXXVIII.

241. Invenire radium oſculatorem ad determinatum curvæ punctum.

Sufficiatur in formula radii oſculatoris finita valor ipſius x determinato alicui puncto conveniens, & problema ſolvetur, ut patet.

PROBL. LXXIX.

242. Invenire radium oſculatorem in vertice A, ſive lineam AS inter initium evolutæ, & curvæ verticem interceptam.

Quoniam in vertice A eſt $x = o$, deleatur in expreſſione finita jam adinventa radii oſculatoris terminus, in quo occurrit x, & problema confectum erit, ut conſtat.

COROLL. I.

243. Quoniam ergo in caſu parabolæ radius oſculator eſt $= \frac{(4ax + aa)\sqrt{(4ax + aa)}}{2aa}$ deleatur in ipſa terminus $4ax$, & fiet $AS = \frac{1}{2}a$ = ſubnormali, vel dimidio parametri.

Idipſum ex eo etiam generatim patet, quod ſi concipiamus ordinatam BD accedere ad verticem A, & fieri infinite parvam, tunc DE conveniet cum ipſa BE, & erit BE, aut illo caſu AE, radius oſculi. Et quidem ſi ordinata bd fiat infinite parva accedens ad verticem A, cum normalis de, & ſubnormalis be æquales inter ſe fiant, diminuta in infinitum earundem differentia, prope verticem A concurſus normalium curvæ Ae, de determinans centrum circuli oſculantis curvam in vertice A, ſcilicet maximi circuli eidem inſcriptibilis reperietur in puncto e, vel C, quod diſtabit a vertice A intervallo dimidiæ parametri. Fig 58.

Quare

Quare cum ſit $BE = \frac{ydy}{dx}$, ſi in hoc valore ſuſſiciatur valor alterutrius, & alterum ponatur $= 0$, habebitur, ut patet, valor ipſius BE, vel potius AE. Ex: gr: in parabola erit $ydy = \frac{1}{2}\,adx$, & $\frac{ydy}{dx} = \frac{1}{2}\,a = AE$. In hyperbola erit $ydy = adx + xdx$, adeoque $\frac{ydy}{dx} = a + x = a$. In prima parabola cubica idem radius fiet $= \frac{aa}{3y}$; in quadrato quadratica idem erit $= \frac{aaa}{4yy}$, in ſubſequenti quinti ordinis parabola ipſe fiet $= \frac{a^4}{5yyy}$; atque ita porro. In verſoria idem radius erit $= \frac{1}{2}\,a =$ radio circuli genitoris. Cum autem ſubtangens prope verticem ſit dupla abſciſſæ (§. 40.) patet verſoriam fere coincidere cum parabola parametro AB deſcripta, quam idem ipſe circulus pariter in puncto A oſculatur; Quanquam tota hæc parabola cadit intra Verſoriam, ſiquidem illius ordinata æqualis eſſet chordæ AC, hujus vero æqualis AF ſubtenſæ angulo recto ACF, ideoque majori quam ſit AC. Eſt autem $x = \frac{1}{4}\,a$ in puncto flexus contrarii Verſoriæ, quare ordinata GD in eodem puncto auſert portionem axis $AD = \frac{1}{4}$ diametri, ſcilicet reſpondet foco parabolæ Verſoriam oſculantis.

Fig. 7.

COROLL.

244. Quoniam vero centrum circuli maximi parabolæ inſcriptibilis reperitur in puncto e, vel C diſtante a vertice A per intervallum dimidiæ parametri, patet quod ſi hujuſmodi circulus fuerit circulus A G F radio A C deſcriptus, eadem erit curvatura parabolæ A D in vertice A cum curvatura ejuſdem circuli, quare anguli contactus H A D, H A G, parabolicus nempe, & circularis erunt unius ejuſdemque rationis. Contra vero, quoniam y eſt infinite parva, ſive infinities minor quam a, ipſa quoque $\frac{a}{3}$ erit infinities minor radio oſculi parabolæ cubicæ, adeoque hic radius abſolute infinitus. Et ſane ducatur D E normalis ad quodvis ejuſdem curvæ punctum, erit ex natura curvæ ipſius $BD^3 = AH^2 . AB$, quare $AB : BD = BD^2 : AH^2$, & $3\ AB : BD$, hoc eſt ſubtangens ad ordinatam, vel ordinata B D ad ſubnormalem $BE = 3\ BD^2 : AH^2 = BD^2 : \frac{1}{3} AH^2 = BD : BE = BD^2 : BD . BE$, ergo ſemper $BD . BE = \frac{1}{3} AH^2$; hinc triang. D B E æquale conſtanti quantitati, eruntque ſubnormales omnes reciprocæ ordinatarum, nempe $BD : bd = be : BE$; Eſt autem accedente bd ad verticem, B D infinite major ipſa bd, quare viciſſim erit be infinite major determinata B E; adeoque duæ normales ad curvam prope verticem A, videlicet A e, de convenire non poſſunt, niſi ad infinitam diſtantiam, proindeque centrum circuli oſculatoris e infinite diſtabit a vertice, atque adeo curvatura parabolæ cubicæ A d D eſt infinities minor curvatura parabolæ quadraticæ, & ideo infinities minus recedit a tangente A H, quam recedat parabola quadratica, ſcilicet ang. a tangente, & pa-

S rabola

rabola cubica comprehenſus infinities minor eſt ang. quem eadem tangens cum parabola quadratica comprehendit. Præterea quoniam yy eſt adhuc infinities minus quam ay, & infinitis vicibus infinities minus quam aa, ergo pariter a^4 erit ſecundo gradu infinities minor radio oſculi parabolæ biquadraticæ. Et quidem poſita A d D parabola biquadratica factaque eadem conſtructione oſtendetur fore ſemper ſolidum ex ſubnormali BE in quadratum ordinatæ BD æquale $\frac{1}{4}$ cubi parametri AH, quare ſubnormales ſunt reciprocæ quadratis ordinatarum, ſive ut $BD^2 : bd^2 = be : BE$. Eſt autem $BD^2 : bd^2$ in duplicata ratione finitæ BD ad infinite parvam bd, aut infinitæ ad finitam, ergo radius oſculi hujuſce curvæ ad punctum A erit pluſquam infinitus, ideoque longe infinities minus recedet hujuſmodi parabola a tangente AH, hoc eſt angulum contactus adhuc infinities minorem, quam præcedens, comprehendet. Item quoniam yyy eſt infinities minus ayy, & infinities adhuc minus aay, & iterum infinities minus aaa, erit pariter $\frac{a}{5}$ in tertio gradu infinities minor radio oſculi parabolæ quintæ, quod rurſus eodem proceſſu, quo ſupra, confirmatur. Non abſimili ratione ad parabolas altioris ordinis progrediendo oſtendetur creſcere rationem radii circuli oſculatoris, atque ita infinities minui ſingularum curvaturam, ſemperque magis contingentiæ angulum coarctari (qualiſcumque ſit gradus parabolarum m) in ratione tam multiplicata rationis ſimpliciter infinitæ, quam multiplex eſt $m-2$ unitatis. Sit enim $AB = x$, $BD = y$, & curvæ æquatio $y^m = a^{m-1} x$, erit $my^{m-1} dy = a^{m-1} dx$, & $\frac{a^{m-1} dx}{my^{m-1}} = dy$, unde ſubnormalis

$n =$

$n = \frac{ydy}{dx} = \frac{a^{m-1}}{my^{m-2}}$; Itaque $ny^{m-1} = \frac{a^{m-1}}{m}$ conſtanti, & $y^{m-2} : a^{m-2} = \frac{a}{m} : n$, & ratio $n : \frac{a}{m} = a^{m-2} : y^{m-2}$ æqualis rationi $a : y$ tam multiplicatæ, quam multiplex eſt $m-2$ unitatis. Unde hujuſmodi parabolæ quemlibet circulum ut ut ampliſſimum intra ſe concludunt, cum arcus earundem ſat magnus AdD ad tangentem propius accedat, ipſique quodammodo adhæreat. Quod ſi fuerit hyperbola AD ejus generis, ut poſita AL $= a$, ſit $yyy = (a+x)^2 x = aax + 2axx + xxx$, quoniam $3yydy = aadx + 4axdx + 3xxdx$, adeoque $dy = aadx + 4axdx + 3xxdx$, erit $n = \frac{ydy}{dx} = \frac{aa + 4ax + 3xx}{3y}$, & ubi $x = 0$ in vertice A, erit ſubnormalis $n = \frac{aa}{3y}$, proindeque infinita, cum $3y$ infinite parva ſit ad finitam a, ut eadem a ad ſubnormalem. Si fuerit $y^4 = (a+x)^3 x = aaax + 3aaxx + 3axxx + x^4$, erit $4yyydy = aaadx + 6aaxdx + 9axxdx + 4xxxdx$, & $dy = aaadx + 6aaxdx + 9axxdx + 4xxxdx$, unde $n = \frac{ydy}{dx} = \frac{aaa + 6aax + 9axx + 4xxx}{4yy}$, & ubi $x = 0$, $n = \frac{aaa}{4yy}$, ſcilicet $4yy : aa = a : n$, quæ erit pluſquam infinita, & ita porro.

245. Anguli contactus circularis ſunt reciproci radiis oſculorum.

Sint duo circuli AGF, AKD, quorum radii ſint AC, AE. Ducatur recta GKM radio AC parallela uſque ad occurſum tangentis, & ponatur $AC = a$, $AE = b$, $AM = y$, $MG = x$, $MK = v$, erit $yy = 2ax - xx$, eademque $yy = 2bv - vv$, quare $2ax - xx = 2bv - vv$. Jam vero ponatur nunc recta MG infinite proxima ipſi AE, evaneſcent profecto xx, & vv utpote infinities minora quadrato ipſius AM jam infinite parvo, ſeu yy, adeoque exterminari poterunt ab æquatione; Reſtabit igitur $2bv = 2ax$, proindeque $x : v = b : a$, ſive $MG : MK = AE : AC$, ſiquidem accedente in infinitum recta MG ad radium AE, eadem MG fit perpendicularis arcubus AG, AK, & haberi poteſt ut arculus circularis radio AM infinite parvo deſcriptus, quo menſurari poſſit quantitas angulorum GAM, KAM.

COROLL. I.

246. Hinc patet quo modo datus angulus contingentiæ circularis MAG in data ratione ſecari poſſit, ſi nempe adſumatur radius AE ad radium AC dati circuli in ratione data, eoque radio deſcribatur circuli arcus AKD.

COROLL. II.

247. Patet præterea quamlibet ſectionem conicam, cujus dimidia parameter ſit AE eundem pariter contingentiæ ang. MAG in data ratione ſecare. Nam quævis curva æquicurva eſt circulo ipſam oſculanti. Cum autem ſectiones conicas oſculetur circulus per verticem deſcriptus diametrum habens æqualem parametro, quælibet conica ſectio cujus dimidia parame-

ter

ter $= AE$ curvaturam habebit circulo AKD ipſam oſculanti congruentem, ita ut prope verticem A altera ab altero diſtingui nequeat, quare & ipſa angulum contactus in eadem ratione ſecabit.

COROLL. III.

248. Hinc quantitates anguli contactus rectæ cum qualibet curva revocari poterunt ad menſuram per reciprocum radii circuli curvam ipſam oſculantis. Eſt enim ang. MAG ad ang. MAK reciproce ut radius hujus AE ad radium prioris AC. Sed iidem ſunt anguli contactus curvarum, quas circuli AG, AK oſculantur, ergo hujuſmodi ang. ſunt reciproce ut radii circulorum oſculantium.

PROBL. LXXX.

249. Invenire æquationem ad evolutam SP parabolæ Apollonianæ.

Ducta ex puncto P recta PR parallela ordinatis ponatur $= z$, & $SR = v$. Quoniam vero $PR =$ *Fig. 55.*
$MG = \frac{4x\sqrt{ax}}{a}$, erit $z = \frac{4x\sqrt{ax}}{a}$. Sed $SR = AM + MR - AS$, & $GP = MR = 2x + \frac{1}{2}a$ (ob ſimilit triang. CMK, CGP) quare $SR = x + 2x + \frac{1}{2}a - \frac{1}{2}a = 3x = v$, unde $x = \frac{v}{3}$. Subſtituatur $\frac{v}{3}$ loco x in valore ipſius z, fiet $z = \sqrt{\frac{16vvv}{27a}}$, & $zz = \frac{16vvv}{27a}$, adeoque $27azz = 16vvv$, quæ eſt

æquatio

æquatio ad curvam quæſitam, ſive evolutam SP. Eſt igitur evoluta parabolæ Apollonianæ parabola cubica ſecunda, cujus parameter rationem habet ad parametrum datæ parabolæ quam habet 27 : 16.

COROLL.

250. Patet autem evolutam integræ parabolæ CAC eſſe curvam PSP ad axem AR utrinque convexam, ita ut punctum S ſit punctum regreſſus.

PROBL. LXXXI.

251. Invenire æquationem ad curvam AGK parabolæ BE parallelam, ſive eadem evolutione condeſcriptam.

Sit latus rectum parabolæ BE $= 2a$, erit ergo
Fig. 59. ſubnormalis DN $= a$; Sit etiam AB, vel EK $= b$, AC $= x$, CK $= y$, erit BC $= x - b$ ſumatur IK infinite parva, quæ erit $= dy$, & ex puncto I erigatur normalis IG, erit ergo IG $= dx$, hinc GK $= V(dx^2 + dy^2)$. Jam vero cum ang. NKG ſit rectus, & recto pariter æquales ang. FEK, FKE, demto communi FKE, erit ang. FEK $=$ IKG, quare triang. FEK, IKG erunt ſimilia, ac proinde GK : KI $=$ KE : EF, $V(dx^2 + dy^2) : dy = b : \frac{bdy}{V(dx^2 + dy^2)} =$ EF, hinc BD erit $= x - b + \frac{bdy}{V(dx^2 + dy^2)}$, adeoque DE $= V\left(2ax - 2ab + \frac{2abdy}{V(dx^2 + dy^2)}\right)$ & CN $= a + \frac{bdy}{V(dx^2 + dy^2)} = \frac{ydy}{dx}$, ergo $ydy = adx + \frac{bdxdy}{V(dx^2 + dy^2)}$

& $\frac{b\,dx\,dv}{ydy - adx} = V[dx^2 + dy^2]$ & $bbdx^2 dy^2 =$ $yydy^4 + yydx^2 dy^2 - 2aydxdy^3 - 2aydydx^3$ $+ aadx^2 dy^2 + aadx^4$. Eſto $dy = \frac{vdx}{a}$, erit

$\frac{bbvvdx^4}{aa} = \frac{yyv^4 dx^4}{a^4} + \frac{yyvvdx^4}{aa} - \frac{2yvvvdx^4}{aa}$

$+ vvdx^4 + aadx^4$, unde $v^4 + aavv - \frac{2aavvv}{y}$

$+ \frac{a^4 vv}{yy} - \frac{aabbvv}{yy} + \frac{a^6}{yy} = 0$; at erat $dx = \frac{vdv}{a}$,

quare $\frac{bbvvdy^4}{aa} = yydy^4 + \frac{vyvvdy^4}{aa} - 2yvdy^4$

$- \frac{2yvvvdv^4}{aa} + vvdy^4 + \frac{v^4 dy^4}{aa}$, unde tandem

$v^4 - 2yvvv + aavv - bbvv + yyvv -$ $2aayv + aayy = 0$.

PROBL. LXXXII.

252. Hyperbolæ aſymptoticæ ſuboſculatricem invenire.

Sit hyperbola aſymptotica ABC, cujus æquatio $aa = xy$, erit $\frac{aa}{y} = x$, & $- \frac{aady}{y} = dx$ po- *Fig. 60.*

ſitaque dx conſtante, $- \frac{aavyddy + 2aaydy^2}{y^4} = 0$,

unde $ddy = \frac{2dy^2}{y}$, quo valore ſuffecto in formula

$\frac{dt^2}{-ddy}$, fiet $DB = \frac{ydx^2 + ydy^2}{-2dy^2}$, quare DE, vel $FG = -\frac{ydy}{2dx} - \frac{ydx}{2dy}$.

Ducatur per punctum T in quo tangens BT ſecat aſymptotum MN recta TP parallela EK occurrens BF productæ in puncto P; ſumaturque BD æqualis dimidio rectæ BP ad alteram aſymptoti plagam, eſt enim ejus valor negativus, vel etiam ſumatur FG æqualis dimidio rectæ TK ad eandem plagam puncti T, ſi ducatur DE axi parallela, vel EG axi normalis, ipſæ ſecabunt rectam BE in puncto quæſito E; Eſt enim $BP = \frac{ydx^2 + ydy^2}{dy^2}$, & $TK = \frac{ydy}{dx} + \frac{ydx}{dy}$.

Fig. 61. Aliter juxta methodum §. 238. Sit AB hyperbola inter aſymptotos LMN, propter $FM = FT$, facto $FB : FM = V : FO = V^2 : FOV = MFB : FM^2$, erit $V^2 : BFM = FOV : FM^2$, ergo ſi V ſumatur $= \sqrt{BFM}$, qui eſt conſtans rectangulum hyperbolæ inſcriptum, erit $FO \cdot V = FM^2$, unde MO parabola, adeoque FR ſubtangens $= \frac{1}{2} FM = \frac{1}{2} FT$. Sed $FT : FR = GD : BE$, quare $BE = \frac{GD}{2}$, demiſſaque EK ad aſymptoton perpendiculari, erit & $FK = \frac{DT}{2}$, quod coincidit ſuperiori conſtructioni quæ eſt HOSPITALII *pag.* 83.

PRO-

PROBL. LXXXIII.

253. Invenire radium oſculi in Ellipſi.

Quoniam ad ellipſim eſt $ayy = abx - bxx$, erit $2aydy = abdx - 2bxdx$, & $dy = \frac{abdx - 2bxdx}{2V(aabx - abxx)}$, poſitaque dx conſtante erit *Fig. 62.*

$$ddy = -\frac{aaabbdx^2}{(4aabx - 4abxx)V(aabx - abxx)},$$

ergo $\frac{dt^3}{-dxddy} = [4aabx - 4abxx + aabb - 4abbx + 4bbxx]V[4aabx - 4abxx + aabb - 4abbx + 4bbxx] : 2aaabb$

si nempe neceſſariæ inſtituantur operationes. Ponatur hæc quantitas $= \frac{vVv}{2aaabb}$. Et quoniam $MY = \frac{y}{dx} V(dx^2 + dy^2)$ (§. 125.) erit, neceſſariis inſtitutis operationibus, eadem $MY = \frac{Vv}{2a}$, adeoque $MY^3 = \frac{vVv}{8aaa}$, & $4MY^3 = \frac{vVv}{2aaa}$. Quare $MP = \frac{vVv}{2a^3bb} = \frac{4MY^3}{bb}$.

Sumatur itaque ad parametrum, & normalem MY quarta continue proportionalis, & hujus quadrupla erit radius oſculi MP.

Eodem prorſus modo invenitur radius oſculi in hyperbola.

COROLL.

254. Conſtat autem quod ſi fiat $x = o$ radius circuli in vertice A ellipſim oſculantis erit $= \frac{b}{2}$.

COROLL.

255. Si a fuerit $= b$, quo caſu ellipſis abit in circulum, fiet $MP = \frac{1}{2} a$, ex quo ſequitur radios omnes evolutæ eſſe inter ſe æquales, adeoque circuli evolutam eſſe ejuſdem centrum.

PROBL. LXXXIV.

256. Invenire radium oſculi in Logiſtica.

Eſto Logiſtica vulgaris A B, cujus ſubtangens C T ſit $= a$, quæ ſubtangens eſſe debet quantitas conFig. 63. ſtans, $EC = x$, $CB = y$. Erit ergo $\frac{ydx}{dy} = a$, adeoque $dy = \frac{ydx}{a}$, & $ddy = \frac{dydx}{a} = \frac{ydx^2}{aa}$, ac proinde $\frac{dt^3}{-dxddy} = \frac{(aa+yy)\sqrt{(aa+yy)}}{-yy}$. Quoniam vero quantitas hæc eſt negativa, id argumento eſt rectam B F ſumendam non eſſe verſus axem.

PROBL. LXXXV.

257. Invenire radium oſculatorem C P in quovis puncto C Cycloidis A C D, cujus circulus genitor A I F.

Fig. 64. Ponatur jam factum quod quæritur, & ex puncto C agatur C E parallela baſi D F occurrens circulo in puncto I, diametro in E, ducanturque chordæ A I, I F,

I F, & radius L I. Sit abſciſſa A E = x, ordinata E C = y, circuli diameter = $2a$, & arcus A I = z. Eſt quidem ob circulum E I = $\sqrt{(2ax - xx)}$ & ex natura curvæ $y = z + \sqrt{(2ax - xx)}$; ergo, ſumtis differentiis, erit $dy = dz + \frac{adx - xdx}{\sqrt{(2ax - xx)}}$. Sed ducta ei infinite proxima ipſi E I, illique ducta normali io, eſt E I : L I = $io : iI$, $\sqrt{[2ax - xx]} : a = dx : \frac{adx}{\sqrt{(2ax - xx)}} = dz$, quare erit $dy = \frac{2adx - xdx}{\sqrt{(2ax - xx)}}$, & $dy^2 = \frac{(2a - x)dx^2}{x}$, poſitaque dx conſtante, erit $ddy = - \frac{adx^2}{x\sqrt{[2ax - xx]}}$. Sufficiatur valor ipſius dy^2, & ddy in formula radii oſculatoris, eritque C P = $2\sqrt{(4aa - 2ax)} = 2$ I F.

Aliter. Sit D C A = 2 A F = a, D C = z, D V = x, V C = y, erit A C = $a - z$, A E = $\frac{1}{2}a - y$; Eſt autem ob curvam D A^2 : A C^2 = A F : A E, ergo habebitur æquatio $y = \frac{2az - zz}{2a}$, quare $dy = \frac{adz - zdz}{a}$, & $dy^2 = \frac{aadz^2 - 2azdz^2 + zzdz^2}{aa}$; unde $dz^2 - dy^2 = dx^2 = \frac{2azdz^2 - zzdz^2}{aa}$, & $dx = \frac{dz\sqrt{[2az - zz]}}{a}$; hinc facta dz conſtante, $ddx = \frac{dz^2 [a - z]}{a\sqrt{[2az - zz]}}$; ac proinde C P, vel P D

 =

$= \frac{dydz}{ddx} = \sqrt{[2az - zz]}$, atqui PM, vel DH $=$ $\frac{dydx}{ddx} - y$, ergo, ſubſtituendo, PM, vel DH $=$ $\frac{2az - zz}{2a}$.

COROLL.

258. Quando z eſt $= a$, nempe quando CPG coincidit ipſi AG, fit DPG $=$ AG $= a$, & DK $=$ FG $= \frac{1}{2} a$; quoniam vero inde patet eſſe $DG^2 : DP^2 = DK : DH$, $aa : 2az - zz = \frac{1}{2} a : \frac{2az - zz}{2a}$, ſequitur evolutam DPG cycloidis eſſe cycloidem ipſi æqualem, & ſimilem.

THEOREMA.

259. Si parabola circa axem BG parametrum habeat æqualem radio circuli oſculabitur Ciſſoidem in puncto O, ubi $BC^2 = NCO$, ſcilicet rectang. abſciſſæ in parametrum.

Sit enimvero Ciſſois EOB, & ſumantur puncta H, D ab extremis æque remota; Cum ſit DE : DM $=$ BD : DA $=$ BD^2 : BDA, & DM, ſeu tL : tK $=$ At, vel DB : tB $=$ BDA : tB^2, erit ex æquo DE : tK $=$ DB^2 : tB^2; quare per tria puncta EKB tranſibit parabola, cujus axis BG, parameter tL. Nam $LtK = tB^2$, MDE $=$ DB^2; ac proinde ſi parabola parametrum habeat æqualem radio circuli oſculabitur Ciſſoidem in puncto O; In aliis enim locis

Fig. 8.

cum

eum ordinatæ tB, vel BD quadratum ſit æquale LtK, vel EDM minori quam tK, vel DE in CN parametrum, ordinatæ parabolæ excedent ordinatas Ciſſoidis, quare cadet parabola tota extra Ciſſoidem. Parabola vero parametri majoris non conveniet cum Ciſſoide niſi in vertice B, parametri vero minoris velut tL ſecabit Ciſſoidem in punctis K, & E ab O æque remotis, ubi quadrata ordinatarum parabolæ æqualia EDM, & KtL.

PROBL. LXXXVI.

260. Invenire punctum, in quo Logiſtica majorem habet curvitatem.

Quoniam in Logiſtica BA, cujus ſubtangens $= a$, ordinata $BC = y$ ducta normali BDG conveniente cum axe in puncto D, & cum TG parallela ordinatæ in puncto G, eſt $BF = DG$, erit ipſa BF radius oſculi, ut facile demonſtrari poteſt. Inveſtigandum itaque eſt ubi radius oſculi BF ſit omnium minimus. Eſt quidem $TC : TB = CB : BD = TG : DG$, nempe $a : V(aa+yy) = y + \frac{aa}{y} : y \frac{V[aa+yy]}{a} + \frac{aV[aa+yy]}{y} = DG = BF$, quare, ubi minimus eſt, erit ejus differentia nulla, ſcilicet erit, neceſſariis inſtitutis operationibus, $y^4 + \frac{aayy}{2} = \frac{a^4}{2}$, & addito utrinque $\frac{a^4}{16}$, $y^4 + \frac{aayy}{2} + \frac{a^4}{16} = \frac{9a^4}{16}$, ac proinde $yy = \frac{aa}{2}$. Quare ubi quadratum ordinatæ BC eſt dimidium quadrati ſubtangentis CT, vel ubi ſubnormalis

Fig. 63.

Digitized by Google

malis $CD = \frac{CT}{2}$ [nam $CB^2 = TCD$] ibi habetur minimus radius oſculi, adeoque maxima curvatura.

SCHOL.

261. Obſervandum hic eſt, quod hujuſmodi radii ſemper minui videantur ad minores ordinatas TH, & ad majores CB creſcere. Quod tamen minime verum eſt; Sunt enim ut cubi tangentium TB diviſi per ordinatas CB; quippe $TD = a + \frac{yy}{a}$, & $TG = y + \frac{aa}{y}$, quorum quadrata $= DG^2 = \frac{y^6 + 3aay^4 + 3a^4yy + a^6}{aayy}$ cujus radix $= \frac{V(yy + aa)^3}{ay} = \frac{V\,TB^3}{TCB} = DG$, & $DG^2 = \frac{TB^6}{TC^2 \cdot CB^2}$; unde DG eſt in directa ratione cubi TB, & reciproca CB.

PROBL. LXXXVII.

262. Invenire radium oſculatorem MP in curva AKMC ejus naturæ, ut ducta ejuſdem normali MP, & ſubtenſa AM, ſit ubique ang. MYC : MAC in data ratione $p : q$. *Fig. 62.*

Ducatur ſubtenſa AK infinite proxima ſubtenſæ AM, & normalis Kx, ſitque AX ſubtenſæ AM perpendicularis, & Ax normalis ſubtenſæ AK, centroque A intervallo AK deſcribatur arculus KL; Erit ergo $KM : ML = XM : MA$, ſed $KM : ML$ eſt in ratione compoſita radiorum MP, MA, & angulor. KPM, KAM, ſive $p : q$, quippe MYC, ſive AYP : $MAC = p : q = KyC$, vel AyP, hoc eſt $AYP + yPY$:

yPY : K Ay, ergo & reliquus YPy ad reliquum KAM $= p : q$, adeoque KM : ML, ſive MX : XA $= p$MP : qMA $= p$MX : qMA, quare pMP $=$ qMX, & MP $= \frac{q}{p}$ MX.

PROBL. LXXXVIII.

263. Invenire puncta ad evolutam curvæ A B C ejus naturæ, ut junctis ramis ad punctum Q ſit ſemper CA : AB $=$ CQ : QB.

Eſto QB $= y$, A$b = x$, & AC $= a$, QC $= b$, erit igitur æquatio $y = \frac{bx}{a}$. Porro ducto arcu bS inter ramos infinite proximos, & tangente bL occurrente normali QL in L, erit Sb : bB $=$ QB : BL, $dy : dx = y : \frac{ydx}{dy}$, loco dy poſito ejus valore, erit tangens BL $= \frac{aydx}{bdx} = \frac{ay}{b} = x$; Quare tangens BL æqualis eſt curvæ AB, & punctum L eſt ad evolutam. Fig. 60.

PROBL. LXXXIX.

264. Determinare radium oſculatorem PC, ubi curva ACc ad diametrum refertur, nempe coordinatæ ang. obliquum conſtituunt.

Sit abſciſſa AE $= x$, ordinata EC $= y$, erit EQ $= dx$, cR $= dy$. Quoniam igitur notus eſt ang. AQc, notus erit & ang. cRH, & nota pariter ratio laterum in triang. cRH. Sit ergo ratio ipſius cR : RH eadem ac ratio $m : n$. Erit ergo $m : n = dy : \frac{ndy}{m} =$ RH, quare CH $= dx + \frac{ndy}{m} = \frac{mdx + ndy}{m}$. Sit præterea R$c$: cH $= m : p$, eritque $m : p = dy :$

$p\,d$

$\frac{pdy}{m} = cH$, hinc $cC^2 = \frac{mmdx^2 + 2mndxdy + nndy^2}{mm} + \frac{ppdy^2}{mm}$. Sed $mm = nn + pp$, quare ſubſtituendo, & extrahendo radicem quadratam, erit $cC = \sqrt{\frac{[mdx^2 + 2ndydx + mdy^2)}{m}}$.

Jam vero ductis tangentibus CT, ct quæ erunt inter ſe infinite proximæ, deſcriptoque arculo infinite minimo TO triang. OTt, HcC erunt ſimilia, ut de ſe conſtat, quare erit $Cc : cH = tT : TO$, nempe $\sqrt{[dx^2 + dy^2]} : dy = -\frac{ydxddy}{dy^2} : -\frac{ydxddy}{dy\sqrt{[dx^2+dy^2]}} = TO$. Eſt autem ob ſectores ſimiles TcO, CPc, $TO : cC = TC : CP$, quare erit

$$\frac{-ydxddy}{dy\sqrt{[dx^2+dy^2]}} : \sqrt{\frac{[mdx^2+2ndydx+mdy^2]}{m}} = \frac{y}{dy}\sqrt{(dx^2+dy^2)} : \frac{(dx^2+dy^2)}{-dxddy}$$

$$\sqrt{\frac{(mdx^2+2ndydx+mdy^2)}{m}} = PC.$$

Simili fere ratione determinatur idem radius oſculator PC, ubi coordinatæ acutum angulum conſtituunt.

PROBL. XC.

265. Invenire formulam generalem radii oſculatoris in curvis, quarum ordinatæ a foco, ſive fixo puncto procedunt. Sit

Sit curva ABCD, cujus ordinatæ FB, FC, FD &c. a foco F, ſive fixo puncto procedant. Sumtis arculis BC, CD infinite parvis ducantur ordinatæ FB, FC, FD, & centro F intervallis FB, FC deſcribantur arculi BE, CG, ductiſque chordis BC, CD indefinite productis ſint normales FH, FK. Producatur prætérea chorda HC, quæ conveniat in puncto I cum ordinata FD producta, & centro C deſcribatur arculus DL; Tandem ducantur CP, DP normales ad curvam ABCD in punctis C, & D. Sit FB = y, CE = dy, BE = dx, & FH = v, ſi centro F deſcribatur arculus HN intervallo FH, erit KN = dv. Eſt autem FN : NH = NH : NM, & eſt FN finita quantitas, & NK quantitas infiniteſima primi ordinis, ergo NM erit quantitas infiniteſima ſecundi, adeoque contemnenda relate ad NK, ergo NK ſumi poterit pro æquali ad ipſam MK, ac proinde MK erit = dv. Et quoniam ſimilia ſunt triangula CBE, CFH, erit CB : CE = CF : CH, $\sqrt{(dx^2 + dy^2)}$ ſive $dt : dy = y : \frac{ydy}{dt}$ = CH = CK. Similia quoque ſunt triang. CKM, CDL, quare CK : KM = CD : LD, $\frac{ydy}{dt} : dv = dt : \frac{dv\,dt^2}{ydy}$. Tandem ob ſimilit. ſectorum PCD, CDL, eſt LD : DC = DC : CP, $\frac{dv\,dt^2}{ydy} : dt = dt : \frac{ydy}{dv}$ = CP. Sed tollere oportet dv. Similia ſunt triang. CBE, CFH, ergo BC : BE = CF : FH, $dt : dx = y : \frac{ydx}{dt} = v$ = FH; Quare, ſumtis differentiis, poſitaque dx conſtante, erit $dv =$

Fig 65.

 dx

$\frac{dx\,dy\,dt^2 - y\,dx\,dy\,ddy}{dt^3}$. Sufficiatur hic valor in formula ſupra inventa, erit $CP = \frac{y\,dt^3}{dx\,dt^2 - y\,dx\,ddy}$.

Si nulla fluxio adſumatur conſtans, fiet $CP = \frac{y\,dt^3}{y\,dy\,ddx + dx\,dt^2 - y\,dx\,ddy}$

Si ſumatur conſtans dy, fiet $CP = \frac{y\,dt^3}{dx\,dt^2 + y\,dy\,ddx}$

Denique ſi conſtans adſumatur dt, erit $ddx = -\frac{dy\,ddy}{dx}$, quo valore ſuffecto loco ipſius ddx, erit $CP = \frac{y\,dx\,dt}{dx^2 - y\,ddy}$, vel ſubſtituto valore ipſius ddy, fiet $CP = \frac{y\,dy\,dt}{dx\,dy + y\,ddx}$.

Si ponatur æquatio differentialis curvæ $dy = v\,dx$, adeoque $ddy = dx\,dv$, erit radius oſculi $= \frac{y\,dt^3}{dx\,dt^2 - y\,dx^2\,dv}$.

Si ponatur æquatio $dx = v\,dy$, erit radius oſculi $= \frac{y\,dt^3}{dx\,dt^2 + y\,dv\,dy^2}$, poſita dy conſtante.

Si ſumatur conſtans dt, erit radius $= \frac{y\,dx\,dt}{dx^2 - y\,ddy}$, vel $= \frac{y\,dy\,dt}{dx\,dy + y\,ddx} = \frac{y\,dt}{dx + y\,dy}$.

COROLL.

COROLL.

266. Si y ponatur infinita, quo caſu ordinatæ pro parallelis haberi poſſunt, & in formulis adinventis deleantur termini, in quibus non adeſt y, emergent formulæ, quas invenimus pro curvis ad axem relatis. Et quidem termini, in quibus non adeſt y evaneſcunt relate ad alios, in quibus eadem y reperitur.

SCHOL.

267. Me non monente apparet, formulam ſuboſculatricis in hoc caſu eſſe $= \dfrac{y\,dx\,dt^2}{dx\,dt^2 + y\,dy\,ddx - y\,dx\,ddy}$.

COROLL.

268. Patet ex dictis, quod cum unica tantum formula reperiatur tum radii oſculatoris, tum ſuboſculatricis in curvis ſive ad axem, ſive ad focum relatis, quælibet curva unica tantum evoluta donabitur.

SCHOL.

269. Inſignis Geometra GUIDO GRANDUS in *Schol. prop. 4. de infinitis infinitorum, & infinite parvorum ordinibus* Cl. VARIGNONIUM carpit, quod vires centrales examinans, aut radios evolutarum inquirens, angulum contingentiæ utpote infinite parvum adſumat velut æqualem angulo infinite parvo ad centrum oſculantis circuli a binis radiis infinite proximis conſtituto, indeque triangula iſoſcelia conſideret, eandemque laterum ad baſim proportionem deducat. Verum enim vero non video quo jure VARIGNONIUS culpari debeat, cum in negotio radiorum evolutarum nil niſi certum adſumat. Eſto enim quælibet curva ABD, cujus radius evolutæ relate ad punctum B quæratur. Sumantur hinc inde a puncto B *Fig. 66.*

datæ curvæ duo arculi æquales BA, BD infiniteſimi, & ex centro C circuli oſculatoris ducantur ad puncta A, & D radii CA, CD, & producatur particula AB in G; adſumit VARIGNONIUS ang. ACB ad centrum eſſe = GBD, unde colligit tres CB, BD, DE eſſe continue proportionales poſita BG = BD; qua quidem in re nihil eſt, ni fallor, dubium, & abſurdum. Si enim quæratur centrum circuli oſculantis curvam in puncto B, idem plane eſt, ac ſi quæratur centrum circuli per tria puncta A, B, D tranſeuntis. Jam vero ſi hujus centrum ponatur eſſe in puncto C, ſequitur ang. ACB, GBD eſſe inter ſe æquales, quippe ang. GBC æqualis eſt duobus internis, & oppoſitis triang. BAC, ſcilicet BAC + ACB. Sed ang. GBC eſt = GBD + DBC, & ang. DBC = ABC = BAC, quare ſi ab ang. GBC auferatur DBC, idem omnino erit ac ſi ab eodem auferatur ang. BAC, quare remanebit ang. GBD = ACB, vel BCD; adeoque BC : BD = BD : DG. Neque vero id repugnat ſententiæ ejuſdem GUIDONIS GRANDI in præfato Scholio. Etenim ſi per punctum B ducatur FE normalis rectæ BC, hæc biſariabit in puncto E baſim GD triang. iſoſcelis GBD; Siquidem ang. FBA, EBD, ob normalem ad curvam BC, ſunt inter ſe æquales; Sed ſunt etiam æquales ang. verticales FBA, GBD, ergo æquales ſunt ang. GBE, DBE, atque adeo GE = ED; ductaque per punctum D recta DH parallela ipſi BE, erit ED = BH. Quare cum recta GD ſit, per nuper demonſtrata, $= \frac{BD^2}{BC}$, erit ED = BH = $\frac{BD^2}{2BC}$, ſcilicet 2BC. BD = *B* D, ED, vel *B* H, ut optime obſervat GRANDUS.

Unde conſtat VARIGNONIUM in determinandis radiis evolutarum nil niſi certum adſumere. Hæc itaque controverſia dirimi non poterit, niſi prius ſtatuatur, uter ang. GBD, an EBD dici debeat

angulus

angulus contactus. Et quidem ſi recta *BG*, quippe quæ eſt productio latuſculi A *B* infinite parvi curvæ A *B* D ſit hujus tangens in puncto *B*, negari non poterit ang. G D *B* contingentiæ angulum dici poſſe.

PROBL. XCI.

270. Invenire radium oſculi in Logarithmica ſpirali. Fig. 67.

Eſto Logarithmica ſpiralis C A *B*, cujus ea eſt proprietas, ut ducta ad quodvis ejus punctum tangente A T, & a polo C ordinata C A, ang. C A T ſit conſtans. Ducatur C *B* infinite proxima ipſi *B* A, ratio ipſius *B* D : D A erit conſtans, quare poſita C A $= y$, arculo A D $= dx$, habebitur æquatio $adx = bdy$ facta igitur dx conſtante, erit $ddy = 0$, ſumtaque formula ſuboſculatricis $\frac{ydxdt^2}{dxdt^2 + ydyddx - ydxddy}$, & neceſſariis inſtitutis operationibus, erit ſuboſculatrix A C $= y$; Quare ducta C E normalis ad A C occurret normali A E in puncto quæſito E ad evolutam. Cum autem ſubnormalis C E ſit $= \frac{ay}{b}$, erit

$$A E = \frac{y\sqrt{(aa + bb)}}{b}.$$

Ducta tangente A T ad punctum A, triang. T E A, E A C ſunt ſimilia, & ang. T A C $=$ C E A, ſed ang. T A C eſt conſtans, quare conſtans erit & ang. C E A, ac proinde evoluta C E erit eadem ipſa Logharithmica ſpiralis C A *B*, ſed ſitu inverſo poſita.

PROBL. XCII.

271. Invenire radium oſculi in ſpirali hyperbolica.

Sit ſpiralis hyperbolica C A *B*, in qua ſubtangens eſt conſtans. Fiant omnia, quæ ſupra erit ejus æquatio $ydx = ady$, quare $\frac{dxdy}{a} = ddy$. Sumta

igi-

igitur formula ſuboſculatricis, in eaque neceſſariis inſtitutis operationibus, fiet ſuboſculatrix $= \frac{y(aa+yy)}{aa}$.

Quoniam vero ſubtangens CT eſt $= a$, & ſubnormalis CE $= \frac{yy}{a}$, erit TE $= \frac{aa+yy}{a}$, ſcilicet quarta proportionalis ad ſubtangentem TC, TE, & CA ſuboſculatricem determinabit. Quare ſi a puncto E ducatur recta EF parallela tangenti AT ſecans ordinatam AC productam in puncto F, erit AF quæſita ſuboſculatrix.

DEF.

Fig. 68. 272. *Subevoluta prima* curvæ AEFG genitæ ex evolutione curvæ *BCD* dicitur curva *BIK*, ex cujus evolutione generari concipitur curva *BCD*. Ejuſdem vero curvæ AEFG *ſubevoluta ſecunda* vocatur curva LMN, ex cujus evolutione generari concipimus curvam *BIK*, ſive ſubevolutam primam curvæ AEFG, & ita porro.

PROBL. XCIII.

273. Determinare formulam pro inveniendis radiis in infinitum ſubevolutarum cujuslibet curvæ AEFG, cujus ordinatæ ſunt inter ſe parallelæ.

Sint arculi EF, CD, quos deſcribunt fila EC, CI per evolutionem infinite parvam curvarum *BCD*, *BIK*, patet triang. EDF, CKD, quæ deſcribuntur ab iiſdem filis quolibet inſtanti eſſe ſimilia inter ſe; Sunt enim rectang. in E, & C, & ang. EDF eſt complementum tum ang. EFD, tum CDK; eademque ratione demonſtratur triang. INK &c. quæ a filis IM &c. ſe exſolventibus deſcribuntur eſſe invicem ſimilia. Erit ergo EF : EC = CD : CI = IK : IM &c. Sed lineola CD eſt differentia radiorum EC, IK differentia radiorum CI, MN differentia radiorum IM, & præterea hujuſmodi lineolæ CD, IK, MN ſunt la-

tera

tera infinite parva curvarum BC, BI, NM, ſi curvæ ipſæ conſiderentur ut polygona ex lateribus numero infinitis, & infinite parvis conſtituta, quare cum EC ſit $= \frac{[dx^2 + dy^2]\sqrt{[dx^2 + dy^2]}}{-dx\,ddy}$, erit $CD =$ $[-3dy\,ddy^2\sqrt{[dx^2 + dy^2]} + dddy[dx^2 + dy^2]\sqrt{[dx^2 + dy^2]}] : dx\,ddy^2$, ac proinde $CI =$ $(3dy\,ddy^2(dx^2 + dy^2)\sqrt{(dx^2 + dy^2)} + dddy[dx^2 + dy^2]\sqrt{[dx^2 + dy^2]}] : dx^2\,ddy^3$, & ſic deinceps. His poſitis ſit.

PROBL. XCIV.

274. Invenire radios in infinitum ſubevolutarum parabolæ.

Eſto parabola AEF, cujus abſciſſa $AO = x$, ordinata $OE = y$ & æquatio $yy = ax$, erit, nenceſſariis inſtitutis operationibus, nimirum ſumtis differentiis, poſitaque dx conſtante, & ſuffectis in ſuperiori problemate valoribus ipſarum dy, ddy, $dddy$ &c. erit, inquam, $CI = \frac{3\sqrt{x}(4x+a)\sqrt{(4x+a)}}{a}$ &c.

Si curva AEF fuerit Cyclois, cujus abſciſſa $= x$, $OE = y$, $AB = 2a$ ac proinde ejus æquatio $dy = \frac{dx\sqrt{[2a - x]}}{\sqrt{x}}$, habebitur $CI = -2\sqrt{2ax}$ &c.

PROBL. XCV.

275. Invenire formulam, cujus ope determinari poſſint radii ſubevolutarum in infinitum in curvis, quarum ordinatæ ex polo procedunt.

Quoniam

Quoniam EF eſt $= \sqrt{(dx^2 + dy^2)}$, $EC = \frac{y(dx^2 + dy^2)\sqrt{(dx^2 + dy^2)}}{dx^3 + dxdy^2 - ydxddy}$, erit $CD = [ydyddy[dx^2 + dy^2] - 3yydyddy^2 + dy[dx^2 + dy^2]^2 + y^2 dddy(dx^2 + dy^2)) : (dx^3 + dxdy^2 - ydxddy)^2$, ac proinde $CI = (-3yyydyddy^2 (dx^2 + dy^2) + (yydyddy + yyydddy)(dx^2 + dy^2)^2 + ydy[dx^2 + dy^2]^3] : [dx^3 + dxdy^2 - ydxddy]^3$, atque ita porro.

PROBL. XCVI.

276. Invenire radios ſubevolutarum in infinitum ſpiralis logarithmicæ.

Eſto AE logarithmica ſpiralis, in qua recta AE procedens a polo ſit $= y$, ejuſque æquatio $adx = bdy$; Et quoniam relatio $dy : dx$ eſt conſtans, ſcilicet quæ $a : b$, deleantur in formulis termini omnes, in quibus adeſt ddy, $dddy$, $ddddy$, eritque $EC = \frac{y}{b}\sqrt{(bb + aa)}$, $CI = \frac{ay}{bb}\sqrt{(bb + aa)}$, $IM = \frac{aay}{bbb}\sqrt{(bb + aa)}$.

Qui plura cupit adeat Cl. MAUPERTUISIUM * ex quo doctrinam hanc de ſubevolutis hauſimus.

* *Memoir. de l'Acad. Roy. an.* 1728.

CAP.

CAP. VI.

De uſu Calculi differentiarum ſecundarum in Cauſticis Catoptricis determinandis.

DEF.

277. SI radii paralleli RA, R*a* &c. a ſole prodeuntes incidant in perimetrum cavam, vel convexam cujuslibet curvæ *BAE*, atque inde ita refleƈtantur, ut non in eundem focum, ſed in aliquam curvam LGHE conveniant, quæ omnes infinite proximorum focos compleƈtatur, & quam iidem reflexi radii perpetuo tangunt, quandoquidem duæ tangentes nihil diſtantes in ipſo contingentiæ punƈto ſe ſe interſecant, curva hæc *cauſtica ex reflexione* nuncupatur. Idem dicas, ſi radii ex dato punƈto promanent. *Fig. 69.*

COROLL.

278. Unde conſtat quod ut natura curvæ cauſticæ determinetur, invenienda ſit natura curvæ, quam tangunt radii omnes, qui in data quapiam curva refleƈtuntur.

PROBL. XCVII.

279. Determinare formulam pro invenienda longitudine radii reflexi AL inter punƈtum incidentiæ A, & punƈtum concurſus L intercepti, ubi radii incidentes ſunt inter ſe paralleli.

Sit abſciſſa $EN = x$, ordinata $NA = y$, $AM = v$, erit $Nn = dx = AC$, $aC = dy$. Biſecetur ang. NAM per reƈtam AP, quæ normalis erit ad curvam A*a*, ut de ſe patet, proindeque erit $NP = \frac{ydy}{dx}$. Cum autem ſit $NA : AM = NP : PM$, erit compo-

nendo $NA : NA + AM = NP : NM$, $y : y + v = \frac{y\,dy}{dx} : \frac{y\,dy + v\,dy}{dx} = NM$. At eſt $AN^2 + NM^2 = AM^2$, ergo $yy + \frac{yydy^2 + 2vydy^2 + vvdy^2}{dx^2} = vv$, hoc eſt $\frac{yydy^2 + 2vydy^2 + yydx^2}{dx^2 - dy^2} = vv$, & $\frac{ydy^2 + ydx^2}{dx^2 - dy^2} = v = AM$. Sufficiatur hic valor loco v in valore ipſius NM, erit $NM = \frac{2ydydx}{dx^2 - dy^2}$; Eſt autem $EN = x$, quare $EM = \frac{2ydydx + xdx^2 - xdy^2}{dx^2 - dy^2}$, ejuſque proinde differentia, poſita dx conſtante, erit $= \frac{dx^5 - dxdy^4 + 2ydx^3\,ddy + 2ydxdy^2\,ddy}{(dx^2 - dy^2)^2} = Mm$. Quoniam vero $AN : NM$, ſive $an : nm = aC : CD$, hoc eſt $y : \frac{2ydydx}{dx^2 - dy^2} = dy : \frac{2dxdy^2}{dx^2 - dy^2} = CD$, erit $AD = \frac{dxdy^2 + dx^3}{dx^2 - dy^2}$. Atqui $AD : Mm = AL : LM$, & dividendo $AD - Mm : AD = AM : AL$, quare neceſſariis inſtitutis operationibus invenietur $AL = \frac{dx^2 + dy^2}{-2ddy}$. Hinc ſi in qualibet propoſita curva

curva E A ſubſtituatur valor ipſius dy, & ddy, pro ut poſtulat natura ejuſdem curvæ, habebitur longitudo radii in quantitatibus communibus, & finitis.

COROLL.

280. Quoniam vero cognito radio A L curva Cauſtica conſtrui poteſt, proinde eadem eſt determinata.

PROBL. XCVIII.

281. In circulo BDA determinare curvam cauſticam BFG radiorum parallelorum, ſcilicet reflexi radii EF longitudinem invenire. *Fig.* 70.

Sit diameter $AB = 2a$, $BI = x$, $IE = y = \sqrt{(2ax - xx)}$, erit $dy = \frac{adx - xdx}{\sqrt{(2ax - xx)}}$; $dy^2 = \frac{aadx^2 - 2axdx^2 + xxdx^2}{2ax - xx}$, & $ddy = -\frac{aadx^2}{(2ax - xx)\sqrt{(2ax - xx)}}$ quare $\frac{dx^2 + dy^2}{-2ddy} = EF = \frac{1}{2}\sqrt{(2ax - xx)} = \frac{1}{2} IE$.

COROLL. I.

282. Hinc patet, quod ut conſtruatur cauſtica in circulo, nempe ut habeantur omnia ejuſdem curvæ puncta, adſumendus eſt radius reflexus æqualis dimidio radii incidentis. Quod ſi radius incidens tranſeat per centrum circuli, punctum F incidet in punctum G, nempe in focum circuli qui focus ejuſdem radium bifariam ſecat.

 COROLL.

Digitized by Google

COROLL. II.

283. Hinc facile eſt hujuſmodi curvam rectificare. Cum enim Cauſtica LHE ſit ſemper æqualis radio reflexo plus incidente, ut mox de qualibet curva generaliter oſtendemus, & curva *FB* ſit ad radium reflexum FE ut 3 : 1, ad incidentem vero IE ut 3 : 2, integra curva GF*B* erit ad radium circuli ut 3 : 2.

SCHOL.

Fig. 69. 284. Nunc autem demonſtrandum eſt curvam cauſticam æqualem eſſe radio reflexo plus radio incidente. Eſto itaque quævis curva *B*AE, in qua ex reflexione radiorum RA, R*a* profluxerit cauſtica EHLK. Certum eſt ex alibi demonſtratis rectam AL eſſe tangentem cauſticæ; ſi ergo evolvatur curva EHL, atque ex ejus evolutione deſcribatur curva EOG, recta *a*L congruet ipſi GL, ubi evolutio pervenerit ad punctum L. Quare centro L, intervallo LA deſcribatur arculus AQ, qui quidem parallellus erit arculo OG, adeoque AO = *a*G, & *a*Q erit differentia ipſius *a*G. Sunt autem triang. *a*CA, *a*QA inter ſe æqualia, nam ang. ad C, & Q ſunt recti, & ang. *na*A = *maB* = Q*a*A, & hypothenuſa *a*A utrinque triang. communis eſt, quare *a*C = *a*Q; at eſt *a*C differentia ipſius *an*, & *a*Q differentia ipſius *a*G, proinde *an* = *a*G, atque adeo cum ipſa curva EHL ſit = LG, erit curva EHL æqualis radio reflexo plus incidente.

PROBL. XCIX.

285. Curvæ cauſticæ GFB naturam determinare.

Fig. 70. Sit ſemicirculus BDA, cujus diameter BA, radius incidens IE, reflexus EF. Deſcribatur centro C, intervallo CG circulus HOG, ductaque recta EC conſtruatur circulus EFO ſecans radium EF in puncto R, erit cauſtica BFG cyclois, cujus immobilis circulus

lus eſt HOG, ejuſque genitor EFO, vertex B, principium vero G. Cum enim ang. incidentiæ IEB ſit æqualis ang. reflexionis RED, erit ſegmentum LBE ſimile ſegmento circulari RE, proindeque EL : ER = AB : EO = 4 : 1; & IE : ER = 2 : 1, = IE : EF, adeoque ER = EF, quare circulus ERO tranſit per punctum contactus F. Eſt autem ob ſegmenta ſimilia arcus LBE = 4 arcubus EF, idemque = 4 arcubus OH, unde arcus EF = arcui OH. Denique quoniam ſemiperipheria HGK eſt = 2 ſemiperipheriis EFO, erit quadrans HOG = EFO, proindeque reliquus arcus OF = arcui reliquo OG, & curva cauſtica GFB eſt cyclois.

COROLL.

286. Quoniam vero cycloides omnes evolutione ſua ſibi ſimiles cycloides deſcribunt, curva genita ex evolutione cauſticæ BFG, erit cyclois.

PROBL. C.

287. In parabola ABb longitudinem radii reflexi BP invenire.

Ponimus hic radios axi normales. Cum enim radii paralleli ad axem poſt reflexionem in foco ejuſdem parabolæ concurrant, patet totam ejuſdem curvæ parabolicæ cauſticam in unicum punctum abire. Eſto igitur parabola ABb, cujus vertex A, axis AD, parameter $= a$ abſciſſa AC $= x$ ordinata CB $= y = \sqrt{ax}$, radii incidentes CB, cb, reflexi vero BD, bD. Erit ergo $dy = \frac{a\,dx}{2\sqrt{ax}}$, $dy^2 = \frac{a\,dx^2}{4x}$, & $ddy = -\frac{a\,dx^2}{4x\sqrt{ax}}$, quare $\frac{dx^2 + dy^2}{-2ddy} = \frac{(a+4x)\sqrt{ax}}{2a} =$ BD, nempe ut dupla parameter ad ſummam parametri, & quadruplæ abſciſſæ, ita ordinata CB ad quæſitam BD;

BD; Eſt autem curva cauſtica æqualis radio reflexo plus radio incidente; Quare curva $AD = \frac{4x\sqrt{ax}+3a\sqrt{ax}}{2a}$.

PROBL. CI.

288. Invenire longitudinem radii reflexi in curva Cycloidali.

Quoniam in Cycloide eſt $dy = \frac{2adx - xdx}{\sqrt{(2ax-xx)}}$ (257.) erit $dy^2 = \frac{(2a-x)dx^2}{x}$, & $ddy = -\frac{adx^2}{x\sqrt{(2ax-xx)}}$, quare ſubſtituendo erit $\frac{dx^2+dy^2}{-2ddy} = \sqrt{[2ax-xx]}$; proindeque radius reflexus æqualis eſt ordinatæ reſpondenti in circulo genitore, adeoque duplus radii reflexi in eodem circulo.

PROBL. CII.

289. Invenire formulam, cujus ope determinari poſſit longitudo radii reflexi, ubi incidentes radii ſunt inter ſe paralleli.

Eſto quævis curva data ABC, datumque poſitione punctum D, ex quo procedant radii incidentes DB, Db in curvam, & quorum reflexi ſint BE, bE. Ex puncto D erigantur rectæ DF, Df normales rectis
Fig. 72. BD, bD, ductaque ex puncto B recta BLG iiſdem parallela ponatur DB, vel $Db = y$, $bL = dy$, $BL = dx$, & $BF = v$. Biſecetur jam ang. DBF recta BM, quæ normalis erit curvæ Bb; Quare erit $dx : dy = DB : BM$, proindeque $DM = \frac{ydy}{dx}$, quare procedendo ut in problemate 97. reperietur $v = \frac{ydy^2 + ydx^2}{dx^2 - dy^2}$, $DF = \frac{2ydxdy}{dx^2 - dy^2}$; Hinc, po-

sita

ſita dx conſtante, invenietur ejuſdem differentia

$$\frac{2ydx^3\,ddy - 2dxdy^4 + 2ydxdy^2\,ddy + 2dx^3\,dy^2}{(dx^2 - dy^2)^2}$$

$= Df - DF$, nempe fK. Sed ſimilia ſunt triang. DBL, DFK, unde $DB : DF = BL : FK$, quare $FK = \frac{2dx^2\,dy}{dx^2 - dy^2}$. Item ſimilia ſunt triang. DBF, FKH, proinde $DB : DF = FK : KH$, $y : \frac{2ydxdy}{dx^2 - dy^2} = \frac{2dx^2\,dy}{dx^2 - dy^2} : \frac{4dx^3\,dy^2}{(dx^2 - dy^2)^2} = KH$, adeoque $fK - HK$, ſive $fH =$

$$\frac{2ydx^3\,ddy + 2ydxdy^2\,ddy - 2dxdy^4 - 2dx^3dy^2}{[dx^2 - dy^2]^2}.$$

Quoniam vero $BD : DF$, ſive $bD : Df = bL : LG$, $y : \frac{2ydxdy}{dx^2 - dy^2} = dy : \frac{2dxdy^2}{dx^2 - dy^2} = LG$, erit $BL + LG$ nempe $BG = \frac{dx^3 + dxdy^2}{dx^2 - dy^2}$. Sed triangul. BEG, HEf ſunt ſimilia, quare $BE : EH$, vel $FE = BG : fH$, & dividendo $BF : BE = BG - fH : BG$, ſive reductione facta ad eundem denominatorem

$$\frac{dx^5 + dxdy^4 - 2ydx^3\,ddy - 2ydxdy^2\,ddy + 2dx^3\,dy^2}{(dx^2 + dy^2)^2} :$$

d x

Digitized by Google

$\frac{dx^3 + dxdy^2}{dx^2 - dy^2}$, hoc eſt dividendo $\frac{dx^2 + dy^2 - 2yddy}{dx^2 - dy^2}$:

$z = BF$, vel v, ſcilicet $\frac{ydy^2 + ydx^2}{dx^2 - dy^2}$: BE, quæ

proinde erit $= \frac{ydy^2 \mp ydx^2}{dx^2 + dy^2 - 2yddy}$.

Si itaque fiat ſubſtitutio valoris ipſius dy, & ddy, vel etiam ipſius dx ex data curvæ æquatione, longitudo radii reflexi determinari facile poterit.

PROBL. CIII.

290. Eſto circulus ABD, & ex puncto D in ejus peripheria aſſumto procedant radii DB, Db, & in eandem peripheriam incidant; ipſorum autem radiorum reflexi ſint BE, bE, quæritur longitudo radii reflexi BE. *Fig.* 73.

Ex puncto D ducatur circuli diameter DA, & ex punctis B, b demittantur ad ipſam normales BF, bf. Sit ſemidiameter $DC = a$, $DF = v$, $FB = z = \sqrt{[2av - vv]}$, erit $fF = bO = dv$, $bB = \frac{adv}{\sqrt{(2av - vv)}}$, $DB = y = \sqrt{2av}$, cujus differentia $bH = \frac{adv}{\sqrt{2av}} = dy$, $BH = \frac{adv}{\sqrt{[4aa - 2av]}} = dx$, $ddy = \frac{2avddv - adv^2}{2v\sqrt{2av}}$;

Eſt autem dx conſtans, ergo $ddx =$

4aa

$\frac{4aaddv - 2avddv + adv^2}{[4a-2v]V(4aa-2av)} = 0$; invenitur itaque $ddy = \frac{aadv^2}{(vv-2av)V2av}$. Si itaque fiant neceſſariæ ſubſtitutiones in formula $\frac{ydy^2 + ydx^2}{dx^2+dy^2 - 2yddy}$, invenietur $BE = \frac{1}{3} V2av = \frac{1}{3} BD$.

PROBL. CIV.

291. Eſto parabola ABb, cujus vertex A, ex quo promanent radii AB, Ab, ejus axis ſit AC, radii vero reflexi BD, bD, quæritur longitudo BD. *Fig. 71.*

Sit parabolæ parameter $= a$, $AC = v$, $CB = z = Vav$, $AB = y = V(vv + av)$, $BO = dv$, $bO = \frac{adv}{2Vav}$, $Bb^2 = \frac{(a+4v)dv^2}{4v}$, $bo^2 = \frac{[4vv + 4av + aa]dv^2}{4vv+4av}$, quare $Bo^2 = \frac{adv^2}{4v+4a} = dx^2 =$ quantitati conſtanti; hinc ejuſdem differentia $\frac{8avdvddv + 8aadvddv - 4adv^3}{[4v+4a]^2} = 0$, quare $ddv = \frac{dv^2}{2v+2a}$; Eſt autem $dy = \frac{2vdv + adv}{V(4vv+4av)}$, ergo $ddy = \frac{-aadv^2 + [2vddv + addv][2vv + 2av]}{[2vv+2av]V[4vv+4av]}$,

Y ſuffectoque

ſuffectoque valore ipſius ddv, erit $ddy = \frac{(aa + 2vv + av)\,dv^2}{(2vv + 2av)\sqrt{(4vv + 4av)}}$, unde tandem

$$\frac{y\,dx^2 + y\,dy^2}{dx^2 + dy^2 - 2y\,ddy} = \frac{(a + 4v)\sqrt{(vv + av)}}{3a} =$$

BD; & cauſtica AD $= \frac{[4a + 4v]\sqrt{[vv + av]}}{3a}$.

CAP. VII.

De uſu Calculi differentiarum ſecundarum in Cauſticis Dioptricis determinandis.

DEF. I.

292. Nomine *refractionis* intelligitur mutatio directionis, quam radius lucis oblique incidens in ſuperficiem aliquam patitur, ubi ſuperficiem ipſam permeavit. Ita ex: gr. ſi radius lucis AD oblique incidat in ſuperficiem BC, & poſtquam eam permea-
Fig. 74. vit, directionem novam DG a recta DH normali ipſi *B*C recedentem, vel directionem DE ad eandem normalem accedentem adſumat, radius DG, vel DE, dicitur radius refractus, eaque vel declinatio, vel mutatio directionis dicitur refractio.

SCHOL. I.

293. Conſtat experientia, quod ſi radius lucis ex medio rariori in denſius ingrediatur, ejuſdem directio DE accedit ad normalem DH, ſi vero ex denſiori in rarius ingrediatur, ejus directio DG ab eadem normali DH recedit, quando vero radius KD normalis eſt ſuperficiei *B*C, ejuſdem directio non mutatur.

SCHOL.

Digitized by Google

SCHOL. II.

294. Patet ex dioptrica certam, ac determinatam eſſe rationem inter ſinum ang. incidentiæ, & ſinum ang. refractionis ex quovis dato medio in aliud quæcumque ſit radii inclinatio. Ita ſi ſemel conſtet ſinum incidentiæ ex aere in Chryſtallum eſſe ſeſquilaterum ſinus refractionis in certa quadam inclinatione data, ſemper in qualibet alia inclinatione radii ſinus incidentiæ, & refractionis eandem rationem 3 : 2 obtinebunt. Si conſtet ſinum incidentiæ ex aere in aquam eſſe ſeſquitertium ſinus refractionis, ſinus incidentiæ, & refractionis eandem ſemper habebunt rationem 4 : 3. Et ſic in aliis mediis.

DEF. II.

295. Si radii refracti non in idem punctum ſed in curvam aliquam conveniant, quæ omnes ſimul infinite proximorum focos complectatur, puncta ſcilicet, in quæ collimant radii refracti, ubi ſcilicet ii ſaltem, qui infinite proximi ſunt, aut revera conveniunt, aut producti convenirent, curvam autem ipſam iidem radii perpetuo tangent, hujuſmodi curva dicetur *Cauſtica ex refractione*.

PROBL. CV.

296. Data qualibet curva ABC, poſitiſque radiis incidentibus EB, *eb* parallelis, quorum refracti ſint BF, *b*F, ratio vero refractionis *m* : *n*, & punctum in quo refracti concurrunt ſit F, invenire formulam, cujus ope determinari poſſit longitudo *B*F. *Fig. 75.*

Ducantur ordinatæ *B*D, *bd* ad axem radio E*B* parallelum, ductaque *B*G normali ad curvam, GH perpendiculari ad radium refractum *B*F, & *B*K axi AG parallela, quæ ſecet *bd* in puncto I, ponatur AD $= x$, D*B* $= y$, DL $= z$, erit D*d* $= dx$, *b*I $=$

 *d*y

dy, $BL = V[yy+zz]$. Et quoniam ang. $bBG =$ ang. IBD, ſunt enim ambo recti, ablato comuni IBG, erit reliquus $bBI =$ reliquo GBD, & ang. ad I, & D ſunt recti, quare triang. bIB, GDB ſunt ſimilia, eritque propterea $BI : Ib = BD : DG$, $dx : dy = y : \frac{ydy}{dx} = DG$. Eſt autem, ob parallelas BK, AG, ang. $BGD =$ ang. incidentiæ KBG, & HBG eſt ang. refractionis, quare ſin. ang. BGD ad ſin. ang. HBG, ſcilicet ſumta BG pro ſinu toto, $BD : GH = m : n$. Sed ob ſimilit. triangul. LGH, LBD eſt $BD : GH = BL : LG$, $m : n = V[yy-zz] : \frac{n}{m} V[yy+zz] = LG$, quare $DL - DG = z - \frac{ydy}{dx} = \frac{n}{m} V[yy+zz]$, & quadrando, & æquationem ordinando fit $zz = \frac{2mmzydydx - mmyydy^2 + nnyydx^2}{[mm-nn]dx^2}$, qua reſoluta, inſtitutaque diviſione per dx, habetur $z = \frac{mmydy + V(mmnnyydy^2 + mmnnyydx^2 - n^4yydx^2)}{mmdx - nndx} = DL$, vel dL. At eſt $bd : dl = bI : IK$, quare $IK = \frac{mmdy^2 + dyV(mmnndy^2 + mmnndx^2 - n^4dx^2)}{mmdx - nndx}$, & $BK = dx + \frac{mmdy^2 + dyV(mmnndy^2 + mmnndx^2 - n^4dx^2)}{mmdx - nndx}$, & $AL = x + \frac{mmydy + V[mmnnyydy^2 + mmnnyydx^2 - n^4yydx^2]}{mmdx - nndx}$, & Ll differentia nempe ipſius $AL = dx + \frac{mmdy^2 + mmyddy +}{mmdx - nndx}$

$$\frac{mmnnydyddy + mmnndy^3 + mmnndx^2dy - n^4dx^2dy^2}{(mmdx - nndx)V(mmnndy^2 + mmnndx^2 - n^4dx^2)}.$$

Eſt autem $BD^2 + DL^2 = BL^2$, quare $BL =$ $V(yy + (m^4yydy^2 + mmnnyydy^2 + mmnnyydx^2$ $- n^4yydx^2 + 2mmydyV[mmnnyydy^2 + mmnnyydx^2$ $- n^4yydx^2)) : (m^4dx^2 - 2mmnndx^2 + n^4$ $dx^2))$. Unde $BK - Ll = (-mmyddy - mmnnydyddy :$ $V(mmnndy^2 + mmnndx^2 - n^4dx^2)) : (mmdx - nndx)$. Sed ob ſimil. triang. BKF, LlF, eſt $BK : Ll = BF :$ LF, & dividendo $BK - Ll : BK = BL : BF$, quare innoteſcet BF, ſi $mmdx^2 - nndx^2 + mmdy^2 + dy$ $V(mmnndy^2 + mmnndx^2 - n^4dx^2)$ ducatur in $V(yy + (m^4yydy^2 + mmnnyydy^2 + mmnndx^2$ $- n^4yydx^2 + 2mmydyV(mmnnyydy^2 + mmnnyydx^2$ $- n^4yydx^2)) : (m^4dx^2 - 2mmnndx^2 + n^4dx^2))$, & productum dividatur per $-mmyddy - mmnydyddy :$ $V[mmdy^2 + mmdx^2 - nndx^2]$ ſi itaque ſufficiatur valor ipſius dx, vel dy, & ddy in qualibet curva data, determinabitur longitudo radii refracti BF, & curva cauſtica proinde conſtruetur.

PROBL. CVI.

297. Invenire longitudinem radii refracti in curvitate vitri parabolica. Patet ex alibi expoſitis ponendum hic eſſe $m = 3$, $n = 2$, quo poſito erit $BF =$ $[5dx^2 + 9dy^2 + dyV[36dy^2 + 20dx^2]] . V[117dy^2$

+ 45

$+45dx^2 + 18dy\,V(36dy^2 + 20dx^2))$: $[-45dxddy - 90dxdyddy : V[9dy^2 + 5dx^2]]$ diviſo tum numeratore, tum denominatore per y. His præmiſſis ſit parabola ABC parametri $= 2a$ erit ergo $x = \frac{yy}{2a}$, adeoque $dx = \frac{ydy}{a}$; Eſt autem dx conſtans, quare ddx, ſive $\frac{yddy + dy^2}{a} = 0$, ac proinde $ddy = -\frac{dy^2}{y}$. Quare ſi loco dx, & ddy ſufficiantur eorum valores, fiet $5dx^2 = \frac{5yydy^2}{aa}$; $dyV[36dy^2 + 20dx^2] = \frac{dy^2}{a}V(36aa + 20yy)$; $V(117dy^2 + 45dx^2 + 18dy\,V(36dy^2 + 20dx^2)) = \frac{dy}{a}V(117aa + 45yy + 18aV[36aa + 20yy]]$; $-45dxddy = 45\frac{dy^3}{a}$; $-\frac{90dxdyddy}{V[9dy^2 + 5dx^2]} = \frac{90dy^3}{V[9aa + 5yy]}$; Si itaque dividatur tum numerator, tum denominator per dy^3, & ordinetur fractio, habebitur $BF = [V[9aa + 5yy]^3 + 18a^3 + 10ayy].V(117aa + 45yy + 18aV[36aa + 20yy]] : (90a^3 + 45aaV(9aa + 5yy))$.

PROBL. CVII.

296. Invenire longitudinem radii refracti in curvitate vitri circulari.

Sit

Sit ABC circulus diametri $2a$, erit ergo $y = V[2ax - xx]$, $dy = \frac{adx - xdx}{V[2ax-xx]}$, & $ddy = - \frac{aadx^2}{(2ax-xx)V(2ax-xx)} = - \frac{aadx^2}{yyy}$; Si ergo loco dy, & ddy eorum valores ſufficiantur, habebitur $9dy^2 = [9aadx^2 - 18axdx^2 + 9xxdx^2]:(2ax-xx) = (9aa - 9yy)dx^2:yy$; $dyV(36dy^2 + 20dx^2) = \left(\frac{adx - xdx}{V(2ax-xx)}\right)V\left(\frac{36aadx^2 - 32axdx^2 + 16xxdx^2}{2ax - xx}\right)$ $= \left(\frac{adx - xdx}{y}\right)V\left(\frac{36aadx^2 - 16yydx^2}{yy}\right) =$ $(adx^2 - xdx^2)V(36aa - 16yy):yy$. Habebitur etiam $V(117dy^2 + 45dx^2 + 18dy\,V(36dy^2 + 20dx^2))$ $= V(117aadx^2 - 72yydx^2 + (18adx^2 - 18xdx^2)V$ $(36aa - 16yy)):y - 45dxddy = 45aadx^3:y^3 -$ $90dxdyddy:V[9dy^2 + 5dx^2] = [90aaadx^3 - 90aaxdx^3]:$ $yyy\,V(9aa - 4yy)$. Suffectis itaque ubique valoribus, diviſoque numeratore, & denominatore per dx^3, reductaque fractione emerget $BF =$

$$\frac{(V(9aa - 4yy)^3 + 18aaa - 8ayy - 18aax + 8yyx)}{45aaV[9aa - 4yy] + 90aaa - 90aax\,V[117aa - 72yy + [18a - 18x]V[36aa - 16yy]]}.$$

SCHOL.

299. Reliquum nunc eſſet, ut longitudinem radii refracti ad calculum revocaremus, ubi radii inciden-

tes

tes a determinato quopiam puncto proficiſcuntur. Sed quando hujuſmodi ſupputatio impl exiſſima evaderet, ac moleſtiſſima, eandem in medio relinquemus, eo magis quod ex dictis eadem ſupputatio erui poſſit. Sed de his conſule Virum Cl. JOHANNEM BERNOULLIUM, ex quo hæc depromta ſunt.

CAP. VIII.

De ſolutione problematum variæ naturæ per calculum differentialem confecta.

PROBL. CVIII.

Fig. 76. 300. Data curva AFH*B*, cujus ordinatæ HO, *B*E ſint majores abſciſſis AO, AE, invenire curvam AMD, cujus portiones AR, AD ſint æquales rectis HO, *B*E.

Fiat ut $TO : V[HO^2 - TO^2]$ ita conſtans C : OQ, & conceſſis quadraturis, de quibus agemus in calculo integrali, ſpatio AOQ ponatur æquale rectang. ex C. RO, erit punctum R ad curvam quæſitam. Ducta ſiquidem FMNV infinite proxima ipſi HOQ, & FK axi parallela, ponatur $C = a$, $AO = x$, $OH = y$, erit $dx : V(dy^2 - dx^2) = a : \frac{aV(dy^2 - dx^2)}{dx} = OQ$, & inſtituta multiplicatione per dx, erit $QON = aV(dy^2 - dx^2) = a$. RL, quare $RL = V[dy^2 - dx^2]$, & $RL^2 + dx^2 = dy^2$, nempe $MR = HK$, atque ita ſemper, quare $AR = HO$.

PROBL.

PROBL. CIX.

301. Invenire curvam ACD, cujus ope quadrari poſſit curva data BLM, ita ut ſingulæ partes ABLF, ABMG æquentur rectang. quantitatis conſtantis AB in curvæ portiones AC, AD. *Fig.* 77.

Eſto $AG = x$, $GM = y$, $GD = v$, $AB = a$, erit ergo FGML, ſeu $ydx = a$, $CD = a\sqrt{(dx^2 + dv^2)}$, & $\frac{ydx}{a} = \sqrt{(dx^2 + dv^2)}$, ſive $\frac{yydx^2}{aa} = dx^2 + dv^2$; quare $yydx^2 = aadx^2 + aadv^2$, & $yydx^2 - aadx^2 = aadv^2$, & tandem $\frac{dx[\sqrt{yy - aa}]}{a} = dv$.

Fiat igitur curva AHK, cujus ordinatæ $GK = \sqrt{(yy - aa)}$ (ſeu $AB^2 + GK^2 = GM^2$, & ſpatium GAK adplicatum ad AB det latitudinem GD, prodibit hinc curva quæſita; erit ſiquidem $av = GAK$, & $adv = KGFH = dx\sqrt{[yy - aa]}$, quare $dv = \frac{dx\sqrt{[yy - aa]}}{a}$, prorſus ut ſupra.

PROBL. CX.

302. Datam curvam ABC in aliam æqualem diverſæ ſpeciei infinitis modis commutare.

Sit abſciſſa AD propoſitæ curvæ $= x$, ordinata $DB = y$, conſtans $AH = 1$, ejuſque æquatio differentialis $mdx = dy$, & deſcripto rectang. HAEK ductaque tangente BT, fiat ut $TD : DB = AH : IL = m$, unde oriatur curva HLM, cujus ſpatium MHK quadrari poterit, nam quævis portio LHI, ſeu $S.\ mdx = y$, quandoquidem $mdx = dy$, ut conſtabit in calculo integrali, quadrari ergo poterit & ſpatium MHAE, ejuſdemque quævis portio $LHAD = x + y$. Biſarientur ordinatæ LD in puncto N, & ſpatium *Fig.* 78.

OADN pariter quadrari poterit. Fiat aliud ſpatium QRS, cujus ordinatæ DR duplum poſſint ipſarum DN, & AV duplum pariter poſſit ipſius AH, completoque rectang. AVZE patet reliquas BX duplum poſſe reliquarum IN; Quoniam ergo $DI^2 + IL^2 = 2$ quadratis DI, IN, æquabuntur illa quadratis DR, RX, & omnia hujuſmodi ſpatia quadrari poterunt; Spatium autem AQRD rectæ AH adplicatum det latitudinem EF, ſpatium vero QVXR ſimiliter adplicatum det latitudinem FG, quæ normaliter ordinetur ad EF, eritque curva EG = curvæ AB. Eſto enim $EF = v$, $FG = z$, $DR = t$, $RX = s$; Et quoniam $dx : dy = 1 : m$, & $dx^2 + dy^2 : dx^2 = 1 + mm : 1$, $dx^2 : dv^2 = 1 : tt$, $dv^2 : dz^2 = tt : ss$, $dz^2 : dz^2 + dv^2 = ss : tt + ss$, quare ex æquo $dx^2 + dy^2 : dz^2 + dv^2 = 1 + mm : tt + ss$. Sed $1 + mm = tt + ss$, ergo $dx^2 + dy^2 = dz^2 + dv^2$.

Fig. 79. Aliter. Sit curva ABC, & ex puncto extremo C ejuſdem baſis agatur recta CE parallela axi AD, & ad quodlibet ejuſdem curvæ punctum B ducatur tangens TBF. Mox ex infinitis adſignabilibus poſita quavis alia curva CGH, cujus ordinata GBK tranſeat per punctum B prioris curvæ, & normalis ejuſdem ſit recta GL, fiat ſemper ut LK : KG ita TF : RN axi AD ordinatam e directo puncti G, patet hinc oriri curvam MNO ſemper variam pro varietate adſumtæ curvæ CGH. Nunc vero differentiæ quadratorum RN, CD ponatur æquale quadratum RS, prodibit inde curva HSV; Denique ſpatium RSVD eidem DC adplicatum det latitudinem RX, orietur curva DXZ propoſitæ ABC æqualis. Eſto enim $CD = a$, $CK = x$, $KG = y$, $CB = v$, $FT = t$, $RX = z$, $RS = s$, erit ex conſtructione LK : KG, adeoque & $dy : dx = t : \surd(aa + ss)$ quare

$dy\surd$

$dy V[aa+ss] = tdx$; Cum autem ſit $t : a = dv : dx$, erit $tdx = adv$, adeoque $adv = dy V[aa+ss]$. Sed ex conſtructione ſumma $\frac{sdy}{a} = z$, proindeque

$\frac{sdy}{a} = dz$, erit $\frac{ssdy^2}{aa} = dz^2$, & $\frac{ssdy^2 + aady^2}{aa}$

$= dz^2 + dy^2$, ſeu $\frac{dy V(ss+aa)}{a} = V(dz^2 + dy^2]$

ſive $a V[dz^2 + dy^2] = dy V[aa+ss]$, oportet ergo $V(dz^2 + dy^2)$ ſcilicet elementum curvæ DX eſſe $= dv$, & curvam $DX = CB = v$.

Alio modo. Ponatur $AB = BC$, & ducatur recta AC, cujus dimidio ſit $= AD$, ductaque DH axi AB parallela ſit ad axem DH curva DF, cujus ordinatæ EF ſint quartæ proportionales poſt ET, *Fig. 80.*

EG, & AB. Eſto nunc $OR = V\frac{1}{2} FO^2$, ſpatium

ODR quadrari poterit, uti & FDO, itidem quadrari poterit ſpatium RDAE, & facta $AL = AC$, ductaque LKN axi AB parallela, quadrabile fiet & ſpatium NLDR. Spatio autem KIDR ſit æquale rectang. CBM, & ſpatio RDAE æquale rectang. ABP, ſi ducantur rectæ MQ, QP ipſis AB, BC parallelæ, quæ conveniant in puncto Q, erit curva BQ æqualis datæ curvæ AS. Sit enimvero $AE = x$, $EG = y$, $AB = 1$, $EF = z$, $KR = V2 - \frac{V2}{2} - \frac{Vzz}{2}$

$= n$, $ER = \frac{V2}{2} + \frac{Vzz}{2} = m$, EK, vel $AC = p =$

$V2$, $BM = v$, $BP = r$; Eſt quidem $dx : dy = 1 : z$, & $zz : 1 = dy^2 : dx^2$, unde $zz+1 : 1 = dy^2 + dx^2 : dx^2$; Et quoniam $KRDL = CBM$, ſcilicet

ſumma ex $ndx = v$, erit $ndx = dv$, & $1 : nn = dx^2 : dv^2$, quare $zz + 1 : nn = dy^2 + dx^2 : dv^2$. Item ob RDAE = ABP, videlicet ſumma ex $mdx = r$, erit $mdx = dr$, & $1 : mm = dx^2 : dr^2$, unde $zz + 1 : mm = dy^2 + dx^2 : dr^2$. Quoniam vero $dv^2 : dx^2 = nn : 1$, & $dx^2 : dr^2 = 1 : mm$, erit $dv^2 : dr^2 = nn : mm$, & $dr^2 : dr^2 + dv^2 = mm : mm + nn$. Itaque $zz + 1 : 1 = dy^2 + dx^2 : dx^2$, $1 : nn = dx^2 : dv^2$, $nn : mm = dv^2 : dr^2$, $mm : mm + nn = dr^2 : dr^2 + dv^2$, quare ex æquo $zz + 1 : mm + nn = dy^2 + dx^2 : dr^2 + dv^2$. Sed $zz + 1 = mm + nn$, nam $m = \frac{z}{\sqrt{2}} + \frac{\sqrt{2}}{2}$, unde $mm = \frac{zz}{2} + z + \frac{2}{4}$, & $n = \sqrt{2} - \frac{\sqrt{2}}{2} - \frac{z}{\sqrt{2}}$, unde $nn = \frac{zz}{2} - z + \frac{2}{4}$, adeoque $mm + nn = zz + 1$, ergo $dy^2 + dx^2 = dv^2 + dr^2$.

SCHOL.

303. Hic adnotaſſe juverit Virum plane ſummum GUIDONEM GRANDUM dum in Theoremate 14. & 20. ſecundæ Appendicis ad quadraturam circuli, & hyperbolæ ſtudet hujuſmodi problema enodare, curvam exhibere longitudine quidem æqualem propoſitæ, ſed ſpecie non diverſam; quæ enim ex ejuſdem methodo profluit curva una eademque eſt cum propoſita, ut idem Vir Cl. fatetur in reſponſione Apologetica ad objectiones Cl. MARCHETTI, ubi elegantem problematis ſolutionem affert. Quod autem juxta methodum ejuſdem GUIDONIS GRANDI, quæ profluit curva ſit

va ſit plane eadem cum propoſita, ſic demonſtrari poteſt. Eſto curva data ABC, quæ ad axem AE referatur, & ſit $AD = x$, $DB = y$. Ducatur per verticem A datæ curvæ recta AG ea quidem lege ut $EG : EA = 2n : nn - 1$, ad quam ex puncto B demittatur normalis BN. Si punctum G cadet inter E, & C, erit $AN = \frac{2ny + (nn-1)x}{nn+1}$, & $BN = \frac{(nn-1)y - 2nx}{nn+1}$. *Fig.* 81.

Sed per Theorema 14. ejuſdem GRANDI eſſe debet $HM = \frac{2ny + (nn-1)x}{nn+1}$, & $ML = \frac{(nn-1)y - 2nx}{nn+1}$, erit $AN = HM$, & $BN = LM$, proindeque figura HLM ſimilis, & æqualis figuræ ABN, quare arcus HL idem plane erit cum arcu datæ curvæ AB.

Quod ſi punctum g cadat extra puncta E, C, & quidem ad eam ab hiſce diſtantiam, ut poſita $Eg : EA = 1 : n$, normalis Bn demiſſa in rectam Ag cadat in partem curvæ convexam, erit $An = \frac{y - nx}{g}$, quæ, ſi normalis Bn cecidiſſet ſuper Ag ad partem curvæ concavam, pariter fuiſſet $= \frac{nx - y}{g}$. Sed utroque caſu ſit $Bn = \frac{ny + x}{g}$, poſita $g = \sqrt{(nn+1)}$, quemadmodum vult GRANDUS; Poſtulat vero prædictum Theorema 20. ut HM ſit $= \frac{y - nx}{g}$, & $LM = \frac{ny + x}{g}$, quare erit $HM = An$, proindeque figura ſimilis erit, & æqualis figuræ ABN, & arcus HL æqualis & ſimilis arcui BA, unde curva HL convexa erit ad axem HM, perinde ac AB eſt convexa relate ad axem Bn.

PROBL. CXI.

304. Invenire naturam curvæ, in quam ſe componit catena ABC ex punctis A, C libere ſuſpenſa.

 Certum

Certum eſt particulam Aa infinite parvam in hoc ſitu a duabus viribus detineri, quarum altera eſt
Fig. 82. pondus ipſius catenæ AaB perpendiculariter trahens per directionem AL, altera vero quædam vis conſtans a horizontaliter trahens per directionem AG, vel BF. Si enimvero ſtatuatur clavus in puncto B loco alterius partis BC, vel in eodem puncto manus adponatur trahens per directionem BF in ſitu ſuo æque conſiſtet catena, ſive incipiat a puncto A, ſive ab a. Poſita ergo $BH = x$, $HA = y$, arcu AaB, ſive catenæ pondere $= z$, erit $z : a = dx : dy$, quare $dy = \frac{adx}{z}$ quæ eſt quæſitæ curvæ æquatio. Quoniam vero dz eſt $= V(dx^2 + dy^2)$, erit $dz = \frac{dxV(aa+zz)}{z}$ & $zdz = dxV(aa+zz)$, & $dx = zdz(aa+zz)^{-\frac{1}{2}}$, quare $x = V(aa+zz)$ ut conſtabit in calculo integrali, utque conſtat ex differentiis in utroque æquationis membro ſumtis, foret ſiquidem $dx = \frac{zdz}{V(aa+zz)}$; unde $xx = aa+zz$, & $zz = xx - aa$, hoc eſt $z = V[xx - aa]$; ac proinde $dy = \frac{adx}{V[xx-aa]}$.

COROLL. I.

305. Ubi $z = 0$, fit $x = a$, & origo ipſarum x diſtat ab infimo catenæ puncto B quantitate a. Si $x = 2a$, fit $z = V3aa$, ubi $z = a$, fit $2aa = xx = 2zz$, & $x = zV2$.

COROLL. II.

Fig. 83. 306. Si fuerit hyperbola æquilatera BbE, cujus centrum C, poſita $BC = a$, $CK = x$, erit $Kb = V[xx - aa] = z$, eſtque $Mb = CK = x = BM = V(zz$

Digitized by Google

$\sqrt{(zz+aa)}$, ergo $zz+aa = xx$; Sed $= Ba+aa$, quare $Ba = z = Kb$, unde ordinata hyperbolæ = curvæ Ba.

PROBL. CXII.

307. Invenire vim centralem in qualibet curva.

Eſto curva $AEa = z$, in cujus tangentem ED ducta ſit ex centro virium C perpendicularis $CD = p$. Agatur etiam recta Ca infinitæ proxima ipſi CE, quæ occurrat tangenti productæ in puncto e, erit vis centralis V proportionalis ipſi ae, ſumto elementari tempore motus dt conſtanti, hoc eſt $= 1$. Sit autem $EC = x$, eſt quidem tempus ut area circa centrum C deſcripta, adeoque dt ut ſector CEA, vel ut ejus duplum $= pdz$, ductoque radio circuli oſculatoris $ER = R$, & ab parallela ipſi CD, fiet $ab = \frac{dz^2}{2R}$. Cum itaque ſit $p : x = (CD : CE$, vel Ca, ſive $Ce)$ $ab : ae$, nempe ut $\frac{dz^2}{2R} : V$, erit $V = \frac{xdz^2}{2pR}$, ſive $= \frac{xppdz^2}{2pppR}$. Sed $pdz = 1$, unde $ppdz^2 = 1$, adeoque $V = \frac{x}{2pppR}$; Scilicet in omni curva vis centralis eſt in ratione directa diſtantiæ x a centro Virium, & reciproca radii oſculatoris r, ac reciproca triplicata normalis p ex eodem centro demiſſæ. Fig. 84.

Aliter. Sit AEa orbita a mobile deſcribenda, & punctum C centrum virium, Ea arcus infiniteſimus curvæ, & Ee ejus tangens in puncto E, ad quam ex puncto C ducatur normalis CD, & agatur recta CA, quæ producatur uſque ad punctum e occurſum tangentis, ductaque EF normali ipſi Ca, & ab parallela ipſi CD, patet quod ſi ER fuerit radius oſculi in E, erunt tres ER, Ea, ba continue proportionales,

tionales, adeoque $ba = \frac{Ea^2}{R}$, & ob ſimilit. triang. cba, & aDC, vel EDC, erit $ea = \frac{EC \cdot Ea}{ER \cdot DC}$, atque ea eſt ut vis centralis V, & quadratum temporis conjunctim, tempus autem ut rectang. $Ea \cdot DC$, quod eſt duplum triang. ECa, quare cum ſit V ut ea directe, & quadratum temporis reciproce, erit V ut $\frac{EC \cdot Ea}{ER \cdot DC}$ directe, & $Ea^2 \cdot DC^2$ reciproce, hoc eſt $V = \frac{EC}{ER \cdot DC^3} = \frac{lx}{rppp}$.

Alio modo. Sit $EF = dy$, $Ea = ds$, celeritas mobilis ſuper curva in $E = c$, erit per nuper demonſtrata $cc = \frac{Vrdy}{ds}$; Jam vero cum celeritas mobilis ſit directe ut ſpatium percurſum $Ea = ds$ & tempus reciproce, quod eſt ut 2. triang. $ECF = xdy$, erit $cc = \frac{ds^2}{xxdy^2} = \frac{VRdy}{ds}$ vel $V = \frac{ds^3}{Rxxdy^3}$, adeoque ſubſtituendo loco ds^3, & dy^3 ipſarum proportionales xxx, & ppp, fiet $V = \frac{x}{Rppp}$.

PROBL. CXIII.

308. Super radio CB quadrantis circuli AEB ſit ſemicirculus CDB, & ducto quovis radio CE ponatur arcus AE ad arcum BF in ſubdupla ratione, ductaque FD axi AC parallela centro C intervallo CD deſcribatur arcus DH occurrens in puncto H radio CE, & hoc

Fig. 85.

hoc ſemper fiat, habebitur AHC tranſiens per omnia puncta H, quæ ſi circa axem AC rotetur curvam ſuperficiem progignet. Quæritur intentio luminis in C poſiti in quovis puncto ejuſdem ſuperficiei.

Ponatur $CB = 1$, CD, vel $CH = x$, erit CG $= xx$, ejuſque quadratum $= x^4$, & quadratum ipſius $GF = 1 - x^4$; Eſt autem elementum $CG = 2xdx$ ad elementum arcus $BF = FG : AC = V[1 - x^4] : 1$, quare elementum prædicti arcus $= \frac{2xdx}{V[1 - x^4)}$, & elementum ſubdupli arcus $AE = \frac{xdx}{V[1 - x^4]}$. At eſt EC : CA [$= 1:x$] ita elementum arcus AE ad elementum arcus concentrici radio CH deſcripti, ſcilicet binis ramis infinite proximis intercepto prope punctum curvæ H, quod quidem elementum $= \frac{xxdx}{V(1 - x^4)}$, ejuſque quadratum $= \frac{x^4 dx^2}{1 - x^4}$, addito quadrato elementi CH, ſeu dx^2, fiet quadratum elementi curvæ $CH = \frac{dx^2}{1 - x^4}$, & ipſum elementum curvæ $= \frac{dx}{V(1 - x^4)} = KO$, eademque ad KP ($= \frac{xxdx}{V(1 - x^4)}$) $= 1 : xx$; Sed intentio luminis in O ad intentionem ejuſdem in A rationem habet compoſitam ex $AC^2 : CO^2$ ($= 1 : xx$) & ex ſinu ang. KOP ad ſinum totum $= KP : KO = xx : 1$, qua-

re

re intentio in puncto O ad intentionem in A rationem habet ex reciprocis compoſitam, ſeu æqualitatis 1 : 1. Ac proinde lumen in *B* poſitum eadem intentione illuſtrabit ſuperficiem curvam ex rotatione curvæ A H *B* prodeuntem.

PROBL. CXIV.

309. Cadat mobile ex A verſus centrum gravium C, & ſit A D E ſcala temporum integralium, A G K ſcala velocitatum, invenire rationem vis centrifugæ ad vim gravitatis in puncto *B*, poſito quod in eodem puncto velocitate ex caſu A *B* concepta mobile circulum *B L* radio C *B* deſcriptum deſcribere cogatur.

Sit abſciſſa communis A *B* $= x$, ordinata *B* K $= v$, ordinata *B* D $= t$, & A C $= a$; Ducatur etiam tangens D T ad punctum D, & ipſi *B* D infinite proxima *bd*, & *d* F, G H axi parallelæ. Patet ex mechanicis, quod mobile tempore caſus B D æquabiliter percurreret ſubtangentem *B* T, adeoque tempore F D $= dt$ percurreret æquabiliter ſpatium *d* F, ſeu *b B* $= dx$ eadem velocitate *B* K $= v$; poſita itaque *B* M $=$ *Bb* $= dx$, ductaque uſque ad circulum M L axi parallela, quoniam tempore *dt* mobile percurreret in tangente, vel in circumferentia ſpatium *B* M, ſeu *B* L $= dx$, vim haberet centrifugam, quæ per ipſam M L exponeretur & eſt M L $= \frac{BM^2}{2BC}$, proindeque $= \frac{dx^2}{2a - 2x}$. At ſi mobile pergeret moveri in recta *B* C, eodem tempore *dt* velocitate v jam adulta conficeret utique aliud ſpatium $= dx$, ſed agente gravitate, & ſubminiſtrante incrementum velocitatis naſcentis dv efficeret ſpatiolum dimidium quartæ proportionalis ad v, dv, & dx; nam ut æquabiliter tempore dt velocitate v fit ſpatium dx, velocitate dv fieret æquabiliter ſpatium proportionaliter minus nempe $\frac{dx dv}{v}$. Verum quia non totum incrementum dv ab

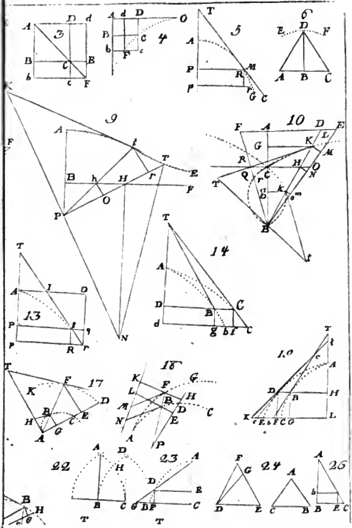
initio

initio temporis viguit in mobili, ſed paulatim acquiſitum fuit, fieret dumtaxat $\frac{1}{2}$ talis ſpatioli; Itaque ſpatiolum ab applicatione gravitatis genitum eſt $\frac{dx\,dv}{2v}$; huic autem proportionatur vis ipſa gravitatis; ergo vis centrifuga in circulo BL ad vim gravitatis in diſtantia $BC = \frac{dx^2}{2a-2x} : \frac{dx\,dv}{2v} = v\,dx : (a-x)\,dv = \frac{v\,dx}{dv} : a-x$, ut ſubtangens ſcalæ velocitatis BC ad diſtantiam centri.

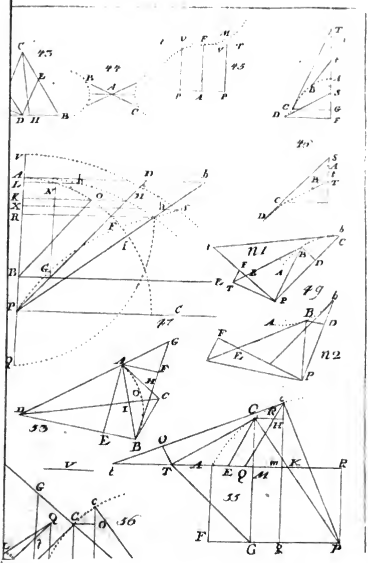
FINIS.

Digitized by Google

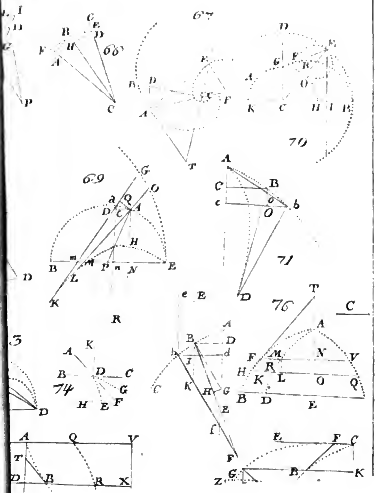
Digitized by Google



Digitized by Google



Digitized by Google



34
G.
37.

Digitized by Google

Digitized by Google